AN ADDENDUM TO "THE THEORY OF IMPLICIT OPERATIONS"

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Abstract

In this addendum to [4], we provide a pair of counterexamples relevant to the theory of implicit operations. More precisely, we exhibit a pp expansion of a variety that fails to be a variety (although it is a quasivariety). Furthermore, we construct a sequence of varieties possessing a nonequational congruence preserving Beth companion.

1. Introduction

An n-ary *partial function* on a set X is a function $f\colon Y \to X$ for some $Y \subseteq X^n$. In this case, the set Y will be called the *domain* of f and will be denoted by $\mathsf{dom}(f)$. This notion can be extended to classes of algebras as follows. An n-ary *partial function* on a class of algebras K is a sequence $\langle f^{\boldsymbol{A}} : \boldsymbol{A} \in \mathsf{K}\rangle$, where $f^{\boldsymbol{A}}$ is an n-ary partial function on A for each $\boldsymbol{A} \in \mathsf{K}$. By a *partial function* on K we mean an n-ary partial function on K for some $n \in \mathbb{N}$. When f is a partial function on K and $\boldsymbol{A} \in \mathsf{K}$, we denote the $\boldsymbol{A}$-component of f by $f^{\boldsymbol{A}}$. Lastly, throughout this note by a *formula* we mean a first order formula.

Definition 1.1. A formula $\varphi(x_1, \dots, x_n, y)$ with $n \geqslant 1$ in the language of a class of algebras K is said to be *functional* in K when for all $\boldsymbol{A} \in \mathsf{K}$ and $a_1, \dots, a_n \in A$ there exists at most one $b \in A$ such that $\boldsymbol{A} \vDash \varphi(a_1, \dots, a_n, b)$.

In other words, φ is functional in K when

$$\mathsf{K} \vDash (\varphi(x_1, \dots, x_n, y) \sqcap \varphi(x_1, \dots, x_n, z)) \to y \approx z.$$

In this case, φ induces an n-ary partial function $\varphi^{\boldsymbol{A}}$ on each $\boldsymbol{A} \in \mathsf{K}$ with domain

$$\mathsf{dom}(\varphi^{\boldsymbol{A}}) = \{\langle a_1, \dots, a_n\rangle \in A^n : \text{there exists } b \in A \text{ such that } \boldsymbol{A} \vDash \varphi(a_1, \dots, a_n, b)\},$$

defined for every $\langle a_1, \dots, a_n\rangle \in \mathsf{dom}(\varphi^{\boldsymbol{A}})$ as $\varphi^{\boldsymbol{A}}(a_1, \dots, a_n) = b$, where b is the unique element of A such that $\boldsymbol{A} \vDash \varphi(a_1, \dots, a_n, b)$. Consequently,

$$\varphi^{\mathsf{K}} = \langle \varphi^{\boldsymbol{A}} : \boldsymbol{A} \in \mathsf{K}\rangle$$

is an n-ary partial function on K.

Definition 1.2. An n-ary partial function f on a class of algebras K is said to be

(i) *defined by a formula* φ when φ is functional in K and $f = \varphi^{\mathsf{K}}$;
(ii) *implicit* when it is defined by some formula;

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(iii) an *operation* of K when for each homomorphism $h\colon \boldsymbol{A} \to \boldsymbol{B}$ with $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{K}$ and $\langle a_1, \dots, a_n \rangle \in \mathsf{dom}(f^{\boldsymbol{A}})$ we have $\langle h(a_1), \dots, h(a_n) \rangle \in \mathsf{dom}(f^{\boldsymbol{B}})$ and

$$h(f^{\boldsymbol{A}}(a_1, \dots, a_n)) = f^{\boldsymbol{B}}(h(a_1), \dots, h(a_n));$$

(iv) an *implicit operation* of K when it is both implicit and an operation of K.

We denote the class of implicit operations of K by $\mathsf{imp}(\mathsf{K})$.

In elementary classes, implicit operations admit the following description (see [4, Thm. 3.9]).

Theorem 1.3. *Let f be a partial function on an elementary class K. Then f is an implicit operation of K if and only if it is defined by an existential positive formula.*

Example 1.4 (Monoids)**.** A typical example of an implicit operation of the variety K of monoids arises from the idea of "taking inverses". More precisely, for every $\boldsymbol{A} \in \mathsf{K}$ let $f^{\boldsymbol{A}}$ be the unary partial function on A with

$$\mathsf{dom}(f^{\boldsymbol{A}}) = \{a \in A : a \text{ is invertible}\}$$

defined for every $a \in \mathsf{dom}(f^{\boldsymbol{A}})$ as

$$f^{\boldsymbol{A}}(a) = \text{the inverse of } a.$$

Then $\langle f^{\boldsymbol{A}} : \boldsymbol{A} \in \mathsf{K} \rangle$ is an implicit operation of K. ⊠

Although concrete examples of implicit operations have long been known, the theory of implicit operations received its first systematic treatment in [4]. In this note, we exhibit two counterexamples relevant to the general theory of implicit operations. For this, we assume familiarity with the concepts and notation of [4], as well as with the basics of the theory of Heyting algebras (see, e.g., [1, Ch. IX]).

2. A variety with a pp expansion that is a proper quasivariety

Consider the linearly ordered Heyting algebra $\boldsymbol{C}_8$ with universe

$$0 < a_1 < a_2 < \cdots < a_6 < 1.$$

We consider the algebra $\boldsymbol{A}$ obtained by endowing $\boldsymbol{C}_8$ with a constant for the element a_5 as well as with a pair of binary operations $x + y$ and $x * y$ and a pair of unary operations $\Box x$

and $\Diamond x$ defined for every $a, b \in A$ as follows:

$$a + b = \begin{cases} a_6 & \text{if } a = 0 \text{ and } b \in \{a_6, 1\}; \\ a_5 & \text{if } a = 0 \text{ and } b = a_3; \\ a_2 & \text{if } (a = 0 \text{ and } b \notin \{a_3, a_6, 1\}) \text{ or } (a \neq 0 \text{ and } b \neq a_1); \\ a_1 & \text{if } a \neq 0 \text{ and } b = a_1; \end{cases}$$

$$a * b = \begin{cases} 1 & \text{if } a = a_4 \text{ and } b = a_6; \\ 0 & \text{otherwise}; \end{cases}$$

$$\Box a = \begin{cases} 1 & \text{if } a = a_5; \\ 0 & \text{otherwise}; \end{cases}$$

$$\Diamond a = \begin{cases} 1 & \text{if } a \in \{0, a_6, 1\}; \\ a_1 & \text{if } a \in \{a_1, a_2\}; \\ a_3 & \text{if } a \in \{a_3, a_5\}; \\ a_5 & \text{if } a = a_4. \end{cases}$$

Our aim is to prove the following.

Theorem 2.1. *The variety $\mathbb{V}(\boldsymbol{A})$ has a pp expansion that is a proper quasivariety and is not congruence preserving.*

Proof. By [4, Thm. 12.9] every congruence preserving pp expansion of a variety is a variety. So, it is sufficient to show that $\mathbb{V}(\boldsymbol{A})$ has a pp expansion that is a proper quasivariety. The proof proceeds through a series of claims. First, observe that $A - \{a_4\}$ is the universe of a subalgebra $\boldsymbol{A} - \{a_4\}$ of $\boldsymbol{A}$.

Claim 2.2. *We have $\mathbb{S}(\boldsymbol{A}) = \{\boldsymbol{A}, \boldsymbol{A} - \{a_4\}\}$.*

Proof of the Claim. As $\boldsymbol{A} - \{a_4\}$ is a subalgebra of $\boldsymbol{A}$, it suffices to prove the inclusion $\mathbb{S}(\boldsymbol{A}) \subseteq \{\boldsymbol{A}, \boldsymbol{A} - \{a_4\}\}$, which amounts to $\mathsf{Sg}^{\boldsymbol{A}}(\emptyset) = A \smallsetminus \{a_4\}$. First, observe that $\mathsf{Sg}^{\boldsymbol{A}}(\emptyset)$ contains the interpretations $0, a_5$, and 1 of the constants. As

$$0 + 1 = a_6, \quad 1 + 0 = a_2, \quad \Diamond a_2 = a_1, \quad \text{and} \quad \Diamond a_5 = a_3,$$

we conclude that $\mathsf{Sg}^{\boldsymbol{A}}(\emptyset)$ contains a_1, a_2, a_3, and a_6 as well. Hence, $\mathsf{Sg}^{\boldsymbol{A}}(\emptyset) = A \smallsetminus \{a_4\}$. ⊠

Claim 2.3. *Let $\boldsymbol{C} \in \mathbb{V}(\boldsymbol{A})$ be a finite nontrivial chain with second largest element a. Then $\boldsymbol{C}$ is subdirectly irreducible with monolith $\mathsf{Cg}^{\boldsymbol{C}}(a, 1)$.*

Proof of the Claim. It suffices to show that $\mathsf{Cg}^{\boldsymbol{C}}(1, a)$ is the monolith of $\boldsymbol{C}$. First, observe that $\mathsf{Cg}^{\boldsymbol{C}}(1, a) \in \mathsf{Con}(\boldsymbol{C}) - \{\mathrm{id}_C\}$ because $a < 1$, where 1 is the maximum of $\boldsymbol{C}$. Then consider $\theta \in \mathsf{Con}(\boldsymbol{C}) - \{\mathrm{id}_C\}$. As $\theta \neq \mathrm{id}_C$, there exist distinct $b, c \in C$ such that $\langle b, c \rangle \in \theta$. Since $b \neq c$, we have $b \leftrightarrow b = 1$ and $b \leftrightarrow c \neq 1$, where $x \leftrightarrow y$ is a shorthand for $(x \to y) \wedge (y \to x)$. As a is the second largest element of $\boldsymbol{C}$, this implies $(b \leftrightarrow b) \vee a = 1$ and $(b \leftrightarrow c) \vee a = a$. Together with $\langle b, c \rangle \in \theta$, this yields $\langle 1, a \rangle \in \theta$, whence $\mathsf{Cg}^{\boldsymbol{C}}(1, a) \subseteq \theta$. ⊠

Observe that

$$\theta = \mathrm{id}_{A-\{a_4\}} \cup \{\langle a_6, 1\rangle, \langle 1, a_6\rangle\}$$

is a congruence of $\boldsymbol{A} - \{a_4\}$. Then let

$$\boldsymbol{B} = (\boldsymbol{A} - \{a_4\})/\theta.$$

Claim 2.4. *We have* $\mathbb{V}(\boldsymbol{A})_{\mathrm{SI}} = \mathbb{I}(\{\boldsymbol{A}, \boldsymbol{A} - \{a_4\}, \boldsymbol{B}\})$.

Proof of the Claim. Observe that all $\boldsymbol{A}, \boldsymbol{A} - \{a_4\}$, and $\boldsymbol{B}$ are finite nontrivial chains. Therefore, the inclusion from right to left follows from Claim 2.3.

To prove the inclusion from left to right, observe that the variety $\mathbb{V}(\boldsymbol{A})$ is congruence distributive because it has a lattice reduct (see, e.g., [4, Thm. 7.2]). By Jónsson's Theorem (see, e.g., [3, Thm. 6.8]) and [2, Thm. 5.6(2)] we have $\mathbb{V}(\boldsymbol{A})_{\mathrm{SI}} \subseteq \mathbb{HS}(\boldsymbol{A})$. Together with Claim 2.2, this yields

$$\mathbb{V}(\boldsymbol{A})_{\mathrm{SI}} \subseteq \mathbb{H}(\{\boldsymbol{A}, \boldsymbol{A} - \{a_4\}\}).$$

Therefore, to conclude the proof, it will be enough to show that $\boldsymbol{A}$ is simple and that, up to isomorphism, the only nontrivial homomorphic images of $\boldsymbol{A} - \{a_4\}$ are $\boldsymbol{A} - \{a_4\}$ and $\boldsymbol{B}$.

We begin by proving that $\boldsymbol{A}$ is simple, which means that $\mathsf{Con}(\boldsymbol{A})$ has exactly two elements. In view of Claim 2.3, it suffices to show that $\mathsf{Cg}^{\boldsymbol{A}}(a_6, 1) = A \times A$. Observe that $\langle 1, 0\rangle = \langle a_4 * a_6, a_4 * 1\rangle \in \mathsf{Cg}^{\boldsymbol{A}}(a_6, 1)$. As the lattice reduct of $\boldsymbol{A}$ is a chain with extrema 0 and 1, this guarantees that $\mathsf{Cg}^{\boldsymbol{A}}(a_6, 1) = A \times A$.

Lastly, we prove that, up to isomorphism, the only nontrivial homomorphic images of $\boldsymbol{A} - \{a_4\}$ are $\boldsymbol{A} - \{a_4\}$ and $\boldsymbol{B}$. By the definition of $\boldsymbol{B}$ it will be enough to show that for every $\phi \in \mathsf{Con}(\boldsymbol{A} - \{a_4\})$,

$$\phi \notin \{\mathrm{id}_{A-\{a_4\}}, \theta\} \text{ implies } \phi = (A - \{a_4\}) \times (A - \{a_4\}).$$

Consider $\phi \notin \{\mathrm{id}_{A-\{a_4\}}, \theta\}$. Observe that the definition of θ and Claim 2.3 guarantee that $\theta \subsetneq \phi$. Therefore, there exist $c, d \in A - \{a_4\}$ such that $\langle c, d\rangle \in \phi - \theta$. From the definition of θ it follows that

$$c \neq d \quad \text{and} \quad \{c, d\} \neq \{a_6, 1\}.$$

As $c \neq d$ and the lattice reduct of $\boldsymbol{A} - \{a_4\}$ is a chain, we can assume that $c < d$. From $c < d$, the right hand side of the above display, and the fact that a_6 is the second largest element of $A - \{a_4\}$ it follows that $c < a_6$, whence $c \leqslant a_5$. Consequently,

$$\langle 1, a_5\rangle = \langle a_5 \vee 1, a_5 \vee c\rangle = \langle a_5 \vee (c \to c), a_5 \vee (d \to c)\rangle \in \phi$$

and, therefore, $\langle 1, 0\rangle = \langle \Box a_5, \Box 1\rangle \in \phi$. As before, this yields $\phi = (A - \{a_4\}) \times (A - \{a_4\})$. ⊠

Consider the pp formula

$$\varphi(x, y) = \exists z(x + y \approx \Diamond z).$$

Claim 2.5. *The formula* $\varphi(x, y)$ *defines an extendable implicit operation* f *of* $\mathbb{V}(\boldsymbol{A})$ *such that* $f^{\boldsymbol{A}}$ *is a total function defined for every* $a \in A$ *as*

$$f^{\boldsymbol{A}}(a) = \begin{cases} a_1 & \text{if } a \neq 0; \\ a_3 & \text{if } a = 0. \end{cases}$$

Proof of the Claim. We will show that φ defines an extendable implicit operation f of $\mathbb{V}(\boldsymbol{A})$. The description of $f^{\boldsymbol{A}}$ in the statement will be an immediate consequence of our proof.

In view of [4, Cor. 8.14], it suffices to show that every member of $\mathbb{V}(\boldsymbol{A})_{\text{SI}}$ can be extended to one of $\mathbb{V}(\boldsymbol{A})$ in which $\varphi(x, y)$ defines a total unary function. Recall from Claim 2.4 that $\mathbb{V}(\boldsymbol{A})_{\text{SI}} = \mathbb{I}(\{\boldsymbol{A}, \boldsymbol{A} - \{a_4\}, \boldsymbol{B}\})$. As $\boldsymbol{A} - \{a_4\} \leqslant \boldsymbol{A}$, we have $\mathbb{V}(\boldsymbol{A})_{\text{SI}} \subseteq \mathbb{IS}(\{\boldsymbol{A}, \boldsymbol{B}\})$. Consequently, it will be enough to show that $\varphi(x, y)$ defines a total unary function both on $\boldsymbol{A}$ and $\boldsymbol{B}$.

We begin with the case of $\boldsymbol{A}$. We need to prove that for every $a \in A$ there exists a unique $b \in A$ such that $\boldsymbol{A} \vDash \varphi(a, b)$. To this end, consider $a \in A$. We have two cases: either $a = 0$ or $a \neq 0$. First, suppose that $a = 0$. Since

$$a + a_3 = 0 + a_3 = a_5 = \Diamond a_4,$$

the definition of φ guarantees that $\boldsymbol{A} \vDash \varphi(a, a_3)$. Therefore, it only remains to show that $b = a_3$ for every $b \in A$ such that $\boldsymbol{A} \vDash \varphi(a, b)$. Consider $b \in A$ such that $\boldsymbol{A} \vDash \varphi(a, b)$. Then $a + b = \Diamond c$ for some $c \in A$. As $a = 0$, we have $a + b \in \{a_2, a_5, a_6\}$. Together with $\Diamond[A] = \{a_1, a_3, a_5, 1\}$ and $a + b = \Diamond c$, this implies $a + b = a_5$ From the definition of $+$ it thus follows that $b = a_3$, as desired.

Then we consider the case where $a \neq 0$. Since $a + a_1 = a_1 = \Diamond a_1$, the definition of φ guarantees that $\boldsymbol{A} \vDash \varphi(a, a_1)$. Therefore, it only remains to show that $b = a_1$ for every $b \in A$ such that $\boldsymbol{A} \vDash \varphi(a, b)$. Consider $b \in A$ such that $\boldsymbol{A} \vDash \varphi(a, b)$. Then $a + b = \Diamond c$ for some $c \in A$. As $a \neq 0$, we have $a + b \in \{a_1, a_2\}$. Together with $\Diamond[A] = \{a_1, a_3, a_5, 1\}$ and $a + b = \Diamond c$, this implies $a + b = a_1$ From the definition of $+$ it thus follows that $b = a_1$.

Next we consider the case of $\boldsymbol{B} = (\boldsymbol{A} - \{a_4\})/\theta$. Since $\boldsymbol{A} - \{a_4\} \leqslant \boldsymbol{A}$ the definition of θ guarantees that for every $a, b \in A - \{a_4\}$,

$$\begin{aligned} \boldsymbol{B} \vDash \varphi(a/\theta, b/\theta) \iff &\text{ there exists } c \in A - \{a_4\} \text{ such that} \\ &\text{ either } a +^{\boldsymbol{A}} b = \Diamond^{\boldsymbol{A}} c \text{ or } \{a +^{\boldsymbol{A}} b, \Diamond^{\boldsymbol{A}} c\} = \{a_6, 1\}. \end{aligned} \tag{1}$$

Then let $a \in A - \{a_4\}$. As before, we have two cases: either $a = 0$ or $a \neq 0$. First, suppose that $a = 0$. Since

$$a +^{\boldsymbol{A}} a_6 = 0 +^{\boldsymbol{A}} a_6 = a_6 \text{ and } \Diamond^{\boldsymbol{A}} a_6 = 1,$$

from $\langle a_6, 1\rangle \in \theta$ it follows that

$$a/\theta +^{\boldsymbol{B}} a_6/\theta = a_6/\theta = 1/\theta = \Diamond^{\boldsymbol{B}} a_6/\theta.$$

By the definition of φ this guarantees that $\boldsymbol{B} \vDash \varphi(a/\theta, a_6/\theta)$. Therefore, it only remains to show that $b/\theta = a_6/\theta$ for every $b \in A - \{a_4\}$ such that $\boldsymbol{B} \vDash \varphi(a/\theta, b/\theta)$. Consider $b \in A - \{a_4\}$ such that $\boldsymbol{B} \vDash \varphi(a/\theta, b/\theta)$. Let $c \in A - \{a_4\}$ be the element given by the right hand side of (1). As $a = 0$, we have $a +^{\boldsymbol{A}} b \in \{a_2, a_5, a_6\}$. Together with $\Diamond c \in \Diamond[A - \{a_4\}] = \{a_1, a_3, 1\}$, the right hand side of (1) ensures that $a +^{\boldsymbol{A}} b = a_6$. By the definition of $+$ we obtain $b \in \{a_6, 1\}$. As $\langle a_6, 1\rangle \in \theta$, we conclude that $b/\theta = a_6/\theta$, as desired. Then we consider the case where $a \neq 0$. In this case, $a +^{\boldsymbol{A}} a_1 = a_1 = \Diamond^{\boldsymbol{A}} a_1$. Therefore, $\boldsymbol{B} \vDash \varphi(a/\theta, a_1/\theta)$ by the definition of φ. It only remains to show that $b/\theta = a_1/\theta$ for every $b \in A - \{a_4\}$ such that $\boldsymbol{B} \vDash \varphi(a/\theta, b/\theta)$. Consider $b \in A - \{a_4\}$ such that $\boldsymbol{B} \vDash \varphi(a/\theta, b/\theta)$. As before, let $c \in A - \{a_4\}$ be the element given by right hand side of (1). Since $a \neq 0$, we have $a +^{\boldsymbol{A}} b \in \{a_1, a_2\}$. Together with

$\Diamond c \in \Diamond[A - \{a_4\}] = \{a_1, a_3, 1\}$ and the right hand side of (1), it follows that $a +^{\boldsymbol{A}} b = a_1$. By the definition of $+$ we obtain $b = a_1$, whence $b/\theta = a_1/\theta$. ⊠

By Claim 2.5 the formula φ defines some $f \in \mathsf{ext}_{pp}(\mathbb{V}(\boldsymbol{A}))$. Consider the f-expansion $\mathscr{L}_f$ of $\mathscr{L}_{\mathbb{V}(\boldsymbol{A})}$ obtained by adding a new unary function symbol g_f to $\mathscr{L}_{\mathbb{V}(\boldsymbol{A})}$. Moreover, let M be the pp expansion $\mathbb{S}(\mathbb{V}(\boldsymbol{A})[\mathscr{L}_{\mathcal{F}}])$ of $\mathbb{V}(\boldsymbol{A})$ induced by

f and $\mathscr{L}_f$. To conclude the proof, it only remains to show that M is a proper quasivariety.

First, M is a quasivariety by [4, Thm. 10.3(ii)]. We will prove that M is not a variety, i.e., it is not closed under $\mathbb{H}$. Recall from Claim 2.5 that $f^{\boldsymbol{A}}$ is a total function. Therefore, the algebra $\boldsymbol{A}[\mathscr{L}_{\mathcal{F}}]$ is well defined. Moreover, the definition of $\boldsymbol{A}$ and the description of $f^{\boldsymbol{A}}$ in Claim 2.5 guarantee that $A - \{a_4\}$ is the universe of a subalgebra $\boldsymbol{C}$ of $\boldsymbol{A}[\mathscr{L}_{\mathcal{F}}]$. Then from the definition of M it follows that

$$\boldsymbol{C} \in \mathbb{S}(\boldsymbol{A}[\mathscr{L}_{\mathcal{F}}]) \subseteq \mathbb{S}(\mathbb{V}(\boldsymbol{A})[\mathscr{L}_{\mathcal{F}}]) = \mathsf{M}.$$

Now recall that

$$\theta = \mathrm{id}_{A-\{a_4\}} \cup \{\langle a_6, 1\rangle, \langle 1, a_6\rangle\}.$$

As θ is a congruence of $\boldsymbol{A} - \{a_4\} = \boldsymbol{C}{\upharpoonright}_{\mathscr{L}_{\mathbb{V}(\boldsymbol{A})}}$ which, moreover, is compatible with the new operation $g_f^{\boldsymbol{C}} = f^{\boldsymbol{A}}{\upharpoonright}_C$ by Claim 2.5, we obtain that θ is also a congruence of $\boldsymbol{C}$. We will prove that $\boldsymbol{C}/\theta \notin \mathsf{M}$. As $\boldsymbol{C} \in \mathsf{M}$, this will imply that M is not closed under $\mathbb{H}$, as desired.

Suppose, with a view to contradiction, that $\boldsymbol{C}/\theta \in \mathsf{M}$. By the definition of M there exists $\boldsymbol{D} \in \mathbb{V}(\boldsymbol{A})$ such that $f^{\boldsymbol{D}}$ is total and $\boldsymbol{C}/\theta \leqslant \boldsymbol{D}[\mathscr{L}_{\mathcal{F}}]$. Observe that

$$0 +^{\boldsymbol{A}} 1 = a_6 \ \text{ and } \ \Diamond^{\boldsymbol{A}} 1 = 1.$$

Since $\langle a_6, 1\rangle \in \theta$ and $\boldsymbol{C}{\upharpoonright}_{\mathscr{L}_{\mathbb{V}(\boldsymbol{A})}} = \boldsymbol{A} - \{a_4\} \leqslant \boldsymbol{A}$, this yields

$$0 +^{\boldsymbol{C}/\theta} 1/\theta = a_6/\theta = 1/\theta = (\Diamond^{\boldsymbol{A}} 1)/\theta = \Diamond^{\boldsymbol{C}/\theta} 1/\theta.$$

Together with the definition of φ, this guarantees $\boldsymbol{C}/\theta \vDash \varphi(0/\theta, 1/\theta)$. Since φ is a pp formula and $\boldsymbol{C}/\theta \leqslant \boldsymbol{D}[\mathscr{L}_{\mathcal{F}}]$, from [4, Prop. 8.1] it follows that $\boldsymbol{D}[\mathscr{L}_{\mathcal{F}}] \vDash \varphi(0/\theta, 1/\theta)$. As φ is a formula in $\mathscr{L}_{\mathbb{V}(\boldsymbol{A})}$ and $\boldsymbol{D} = \boldsymbol{D}[\mathscr{L}_{\mathcal{F}}]{\upharpoonright}_{\mathscr{L}_{\mathbb{V}(\boldsymbol{A})}}$, we obtain $\boldsymbol{D} \vDash \varphi(0/\theta, 1/\theta)$. Since φ is the formula defining f and $g_f^{\boldsymbol{D}[\mathscr{L}_{\mathcal{F}}]} = f^{\boldsymbol{D}}$, this yields

$$g_f^{\boldsymbol{D}[\mathscr{L}_{\mathcal{F}}]}(0/\theta) = f^{\boldsymbol{D}}(0/\theta) = 1/\theta.$$

Therefore, $g_f^{\boldsymbol{C}/\theta}(0/\theta) = 1/\theta$ because $\boldsymbol{C}/\theta \leqslant \boldsymbol{D}[\mathscr{L}_{\mathcal{F}}]$. On the other hand, we will prove that

$$g_f^{\boldsymbol{C}/\theta}(0/\theta) = g_f^{\boldsymbol{C}}(0)/\theta = g_f^{\boldsymbol{A}[\mathscr{L}_{\mathcal{F}}]}(0)/\theta = f^{\boldsymbol{A}}(0)/\theta = a_3/\theta \neq 1/\theta,$$

thus obtaining the desired contradiction. The first equality above holds by the definition of a quotient algebra, the second because $\boldsymbol{C} \leqslant \boldsymbol{A}[\mathscr{L}_{\mathcal{F}}]$, the third by the definition of $\boldsymbol{A}[\mathscr{L}_{\mathcal{F}}]$, and the fourth by Claim 2.5. Finally, the inequality at the end of the above display follows from the definition of θ. ⊠

Remark 2.6. The proof of Theorem 2.1 yields that $\theta \in \mathsf{Con}(\boldsymbol{C}{\upharpoonright}_{\mathscr{L}_{\mathbb{V}(\boldsymbol{A})}}) - \mathsf{Con}_{\mathsf{M}}(\boldsymbol{C})$, witnessing that the pp expansion M of $\mathbb{V}(\boldsymbol{A})$ is not congruence preserving. ⊠

3. A nonequational congruence preserving Beth companion

All the examples of Beth companions considered in [4] are induced by implicit operations defined by conjunctions of equations, as opposed to arbitrary pp formulas. Such Beth companions have particularly nice properties. For example, they have an equational axiomatization relative to the original class of algebras (see [4, Thm. 10.4]) and are congruence preserving (see [4, Thm. 12.4]). One might therefore wonder whether every quasivariety K with a Beth companion also has a Beth companion induced by implicit operations defined by conjunctions of equations. This is the case, for instance, when K has the amalgamation property (see [4, Thm. 12.7] and [4, Rem. 11.12(vi)]). Our aim is to show that the previous conjecture fails, even when K is a variety with a congruence preserving Beth companion. Actually, a counterexample can be found among some of the simplest varieties of Heyting algebras, as we proceed to illustrate.

For every cardinal κ let $\boldsymbol{A}_\kappa$ be the unique Heyting algebra whose lattice reduct is obtained by adding a new maximum 1 to the powerset lattice $\langle \mathcal{P}(\kappa); \cap, \cup \rangle$. The implication of $\boldsymbol{A}_\kappa$ is defined by the rule

$$a \to b = \begin{cases} 1 & \text{if } a \leqslant b; \\ b & \text{if } a = 1; \\ (\kappa - a) \cup b & \text{if } a, b \in \mathcal{P}(\kappa) \text{ and } a \nleqslant b. \end{cases}$$

As expected, $\boldsymbol{A}_\kappa$ and the powerset Boolean algebra $\mathcal{P}(\kappa)$ are closely related, in the sense that $\mathcal{P}(\kappa)$ is isomorphic to the algebra obtained by quotienting $\boldsymbol{A}_\kappa$ under the congruence that glues 1 with κ and leaves any other element untouched.

The varieties generated by Heyting algebras of the form $\boldsymbol{A}_\kappa$ form the chain

$$\mathbb{V}(\boldsymbol{A}_0) \subsetneq \mathbb{V}(\boldsymbol{A}_1) \subsetneq \cdots \subsetneq \mathbb{V}(\boldsymbol{A}_n) \subsetneq \cdots \subsetneq \mathbb{V}(\boldsymbol{A}_\omega),$$

where $\mathbb{V}(\boldsymbol{A}_\omega) = \mathbb{V}(\boldsymbol{A}_\kappa)$ for every infinite cardinal κ (see [9]).[1]

Definition 3.1. A pp expansion M of a class of algebras K is said to be

- (i) *equational* when it is faithfully term equivalent relative to K to a pp expansion of K of the form $\mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}}])$ with $\mathcal{F} \subseteq \mathsf{ext}_{eq}(\mathsf{K})$;
- (ii) an *equational Beth companion* of K when it is equational and a Beth companion of K.

The remainder of this section is devoted to showing that for $n \geqslant 3$ the variety $\mathbb{V}(\boldsymbol{A}_n)$ provides a counterexample to the conjecture that every congruence preserving Beth companion of a variety is equational. More precisely, we will establish the next result.

Theorem 3.2. *The following conditions hold for every* $\kappa \in \mathbb{N} \cup \{\omega\}$*:*

- (i) $\mathbb{V}(\boldsymbol{A}_\kappa)$ *has a congruence preserving Beth companion;*
- (ii) $\mathbb{V}(\boldsymbol{A}_\kappa)$ *has an equational Beth companion if and only if* $\kappa \in \{0, 1, 2, \omega\}$*.*

[1] Although we shall not rely on this fact, we remark that these are precisely the nontrivial varieties of Heyting algebras of depth $\leqslant 2$ (see also [4, Exa. 10.18]).

It is known that $\mathbb{V}(\boldsymbol{A}_0), \mathbb{V}(\boldsymbol{A}_1), \mathbb{V}(\boldsymbol{A}_2)$, and $\mathbb{V}(\boldsymbol{A}_\omega)$ have the strong epimorphism surjectivity property (see [11, Thm. 8.1]). Consequently, they are their own Beth companions by [4, Thm. 11.6]. When viewed as Beth companions of themselves, they are obviously equational Beth companions. Moreover, recall that all Beth companions of a quasivariety K are faithfully term equivalent relative to K (see [4, Thm. 11.7]). Consequently, if K has an equational Beth companion, then all Beth companions of K are equational. Hence, in order to prove Theorem 3.2, it will be enough to establish the following.

Theorem 3.3. *For every $n \geqslant 3$ the variety $\mathbb{V}(\boldsymbol{A}_n)$ has a congruence preserving Beth companion that is not equational.*

The proof of Theorem 3.3 proceeds through a series of observations. First, if an algebra $\boldsymbol{A}$ has a lattice reduct, then $\mathbb{V}(\boldsymbol{A})$ is congruence distributive (see, e.g., [4, Thm. 7.2]). Therefore, the following is an immediate consequence of a version of Jónsson's Theorem for finitely subdirectly irreducible algebras (see, e.g., [4, Thm. 2.12]) and [2, Thm. 5.6(2)].

Proposition 3.4. *Let $\boldsymbol{A}$ be a finite algebra with a lattice reduct. Then $\mathbb{V}(\boldsymbol{A})_{\text{FSI}} \subseteq \mathbb{HS}(\boldsymbol{A})$.*

As an application of Proposition 3.4, we obtain a transparent description of $\mathbb{V}(\boldsymbol{A}_n)_{\text{FSI}}$.

Proposition 3.5. *For every $n \in \mathbb{N}$ we have $\mathbb{V}(\boldsymbol{A}_n)_{\text{FSI}} = \mathbb{I}(\boldsymbol{A}_0, \dots, \boldsymbol{A}_n) = \mathbb{IS}(\boldsymbol{A}_n)$.*

Proof. By Proposition 3.4 we have $\mathbb{V}(\boldsymbol{A}_n)_{\text{FSI}} \subseteq \mathbb{HS}(\boldsymbol{A}_n)$. Moreover, by inspection it is possible to check that (up to isomorphism) the finitely subdirectly irreducible members of $\mathbb{HS}(\boldsymbol{A}_n)$ are $\boldsymbol{A}_0, \dots, \boldsymbol{A}_n$. Together with $\mathbb{V}(\boldsymbol{A}_n)_{\text{FSI}} \subseteq \mathbb{HS}(\boldsymbol{A}_n) \subseteq \mathbb{V}(\boldsymbol{A}_n)$, this yields $\mathbb{V}(\boldsymbol{A}_n)_{\text{FSI}} = \mathbb{I}(\boldsymbol{A}_0, \dots, \boldsymbol{A}_n)$. Lastly, the equality $\mathbb{I}(\boldsymbol{A}_0, \dots, \boldsymbol{A}_n) = \mathbb{IS}(\boldsymbol{A}_n)$ is an immediate consequence of the definition of $\boldsymbol{A}_0, \dots, \boldsymbol{A}_n$. ⊠

Corollary 3.6. *For every $n \in \mathbb{N}$ we have $\mathbb{V}(\boldsymbol{A}_n) = \mathbb{Q}(\boldsymbol{A}_n)$.*

Proof. From the Subdirect Decomposition Theorem (see, e.g., [3, Thm. 8.6]) and Proposition 3.5 it follows that

$$\mathbb{V}(\boldsymbol{A}_n) = \mathbb{ISP}(\mathbb{V}(\boldsymbol{A}_n)_{\text{FSI}}) = \mathbb{ISPIS}(\boldsymbol{A}_n) \subseteq \mathbb{Q}(\boldsymbol{A}_n).$$

Since the inclusion $\mathbb{Q}(\boldsymbol{A}_n) \subseteq \mathbb{V}(\boldsymbol{A}_n)$ always holds, we conclude that $\mathbb{V}(\boldsymbol{A}_n) = \mathbb{Q}(\boldsymbol{A}_n)$. ⊠

We will make use of the following properties typical of the Heyting algebras of the form $\boldsymbol{A}_\kappa$ for a cardinal κ. As all of them are immediate consequences of the definition of $\boldsymbol{A}_\kappa$, their proof is omitted. First, observe that $\boldsymbol{A}_\kappa$ has a second largest element (namely, κ) that we denote by e. For every $a, b \in A_\kappa$ we have

$$a \vee b = 1 \iff a = 1 \text{ or } b = 1; \tag{2}$$

$$0 < a \leqslant e \iff a \vee \neg a = e; \tag{3}$$

$$a \in \{0, e, 1\} \iff \neg\neg a = 1; \tag{4}$$

$$(a \neq e \text{ or } a = 0) \iff \neg\neg a = a. \tag{5}$$

We recall that an element a of an algebra $\boldsymbol{B}$ with a bounded lattice reduct is an *atom* when $a \neq 0$ and there exists no $b \in B$ such that $0 < b < a$. To simplify notation, we will make use of the following shorthands for every algebra $\boldsymbol{B}$ with a bounded lattice reduct and $a \in B$:

$$\mathsf{at}(\boldsymbol{B}) = \{b \in B : b \text{ is an atom of } \boldsymbol{B}\};$$
$$\mathsf{at}_{\boldsymbol{B}}(a) = \{b \in \mathsf{at}(\boldsymbol{B}) : b \leqslant a\}.$$

Moreover, for every $\boldsymbol{B} \leqslant \boldsymbol{A}_n$ and $a \in B$ the following holds:

$$\text{if } a \neq 1 \text{ then } a = \bigvee \mathsf{at}_{\boldsymbol{B}}(a); \tag{6}$$

$$\text{if } b \in \mathsf{at}(\boldsymbol{B}), \text{ then either } (b \leqslant a \text{ and } b \nleqslant \neg a) \text{ or } (b \nleqslant a \text{ and } b \leqslant \neg a). \tag{7}$$

We also rely on the following properties that hold in every Heyting algebra. First, for every $a_1, \dots, a_m \in A_\kappa$,

$$\bigwedge_{i=1}^{m} \neg a_i = 1 \iff a_i = 0 \text{ for every } i \leqslant m. \tag{8}$$

Second, for every $a, b \in A_\kappa$,

$$a \leqslant b \iff a \to b = 1; \tag{9}$$

$$a \leqslant b \implies \neg\neg a \leqslant \neg\neg b. \tag{10}$$

Now, fix $n \geqslant 3$. For each positive $m \leqslant n-1$ let $s_{m,n}$ and d denote the terms

$$s_{m,n} = \bigvee_{i=1}^{n+1} z_i^m \quad \text{and} \quad d = x \vee \neg x,$$

where $x, z_1^m, \dots, z_{n+1}^m$ are variables. Then let $\psi_{m,n}(x, y, z_1^m, \dots, z_{n+1}^m)$ be the conjunction of the following formulas:

$$\prod_{i=1}^{n+1} (d(x) \approx d(z_i^m));$$

$$d(x) \vee \neg\neg(x \vee s_{m,n}) \approx y;$$

$$\Big((s_{m,n} \to x) \wedge \bigwedge_{\substack{i,j=1 \\ i \neq j}}^{m+1} \neg(z_i^m \wedge z_j^m)\Big) \vee \Big((s_{m,n} \to \neg x) \wedge \bigwedge_{\substack{i,j=m+2 \\ i \neq j}}^{n+1} \neg(z_i^m \wedge z_j^m)\Big) \approx 1.$$

For each positive $k \leqslant n-1$ let $\gamma_{k,n}(x, y, z_1^1, \dots, z_{n+1}^1, \dots, z_1^k, \dots, z_{n+1}^k, w_1, \dots, w_k)$ be the formula

$$\Big(y \approx \bigvee_{m=1}^{k} w_m\Big) \sqcap \prod_{m=1}^{k} \psi_{m,n}(x, w_m, z_1^m, \dots, z_{n+1}^m)$$

and define

$$\varphi_{k,n}(x, y) = \exists z_1^1, \dots, z_{n+1}^1, \dots, z_1^k, \dots, z_{n+1}^k, w_1, \dots, w_k \gamma_{k,n}.$$

Observe that $\varphi_{k,n}(x, y)$ is a pp formula for every $n \geqslant 3$ and positive $k \leqslant n-1$. We will prove the following.

Proposition 3.7. *For every $n \geqslant 3$, positive $k \leqslant n-1$, and $a, b \in A_n$,*

$$\begin{aligned} \boldsymbol{A}_n \vDash \varphi_{k,n}(a,b) \iff & \text{ either } (a \in \{0,e,1\} \text{ and } b = 1) \\ & \text{ or } (0 < a < e \text{ and } b = 1 \text{ and the number of atoms below } a \text{ is } \leqslant k) \\ & \text{ or } (0 < a < e = b \text{ and the number of atoms below } a \text{ is } \geqslant k+1). \end{aligned}$$

Proof. We begin by proving the implication from left to right. Suppose that $\boldsymbol{A}_n \vDash \varphi_{k,n}(a,b)$. Then there exist $c_1^1, \dots, c_{n+1}^1, \dots, c_1^k, \dots, c_{n+1}^k, d_1, \dots, d_k \in A_n$ such that

$$b = \bigvee_{m=1}^{k} d_m \tag{11}$$

and for every positive $m \leqslant k$ both

$$a \vee \neg a = c_1^m \vee \neg c_1^m = \cdots = c_{n+1}^m \vee \neg c_{n+1}^m; \tag{12}$$

$$d_m = a \vee \neg a \vee \neg\neg\Big(a \vee \bigvee_{i=1}^{n+1} c_i^m\Big) \tag{13}$$

and

$$1 = \Big(\Big(\bigvee_{i=1}^{n+1} c_i^m \to a\Big) \wedge \bigwedge_{\substack{i,j=1 \\ i \neq j}}^{m+1} \neg(c_i^m \wedge c_j^m)\Big) \vee \Big(\Big(\bigvee_{i=1}^{n+1} c_i^m \to \neg a\Big) \wedge \bigwedge_{\substack{i,j=m+2 \\ i \neq j}}^{n+1} \neg(c_i^m \wedge c_j^m)\Big).$$

Together with (2), (8), and (9), the above display yields that for every $m \leqslant k$,

$$\begin{aligned} &\text{either } \Big(\bigvee_{i=1}^{n+1} c_i^m \leqslant a \text{ and } c_i^m \wedge c_j^m = 0 \text{ for all distinct } i,j \text{ with } 1 \leqslant i,j \leqslant m+1\Big) \\ &\text{or } \Big(\bigvee_{i=1}^{n+1} c_i^m \leqslant \neg a \text{ and } c_i^m \wedge c_j^m = 0 \text{ for all distinct } i,j \text{ with } m+2 \leqslant i,j \leqslant n+1\Big). \end{aligned} \tag{14}$$

By the definition of $\boldsymbol{A}_n$ we have two cases: either $a \in \{0, e, 1\}$ or $0 < a < e$. First, suppose that $a \in \{0, e, 1\}$. We need to prove that $b = 1$. To this end, observe that for every $m \leqslant k$,

$$\neg\neg a \leqslant \neg\neg\Big(a \vee \bigvee_{i=1}^{n+1} c_i^m\Big) \leqslant a \vee \neg a \vee \neg\neg\Big(a \vee \bigvee_{i=1}^{n+1} c_i^m\Big) = d_m,$$

where the first inequality holds by (10), the second is straightforward, and the last equality by (13). Since $a \in \{0, e, 1\}$, we have $\neg\neg a = 1$ by (4). Together with the above display, we obtain $d_m = 1$ for every $m \leqslant k$. By (11) we conclude that $b = 1$, as desired.

Next, we consider the case where $0 < a < e$. In this case, $a \vee \neg a = e$ by (3). Therefore, from (12) it follows that $c_i^m \vee \neg c_i^m = e$ for all positive $m \leqslant k$ and $i \leqslant n+1$. By (3) this yields

$$0 < c_i^m \text{ for all positive } m \leqslant k \text{ and } i \leqslant n+1. \tag{15}$$

We have two subcases: either the number of atoms below a is $\leqslant k$ or $\geqslant k+1$. First, suppose that it is $p \leqslant k$. We need to prove that $b = 1$. As $\boldsymbol{A}_n$ has n atoms by definition, the number of atoms below $\neg a$ is $n - p$ by (7). From (14) in the case where $m = p$ it follows that

$$\text{either } \Big(\bigvee_{i=1}^{n+1} c_i^p \leqslant a \text{ and } c_i^p \wedge c_j^p = 0 \text{ for all distinct } i, j \text{ with } 1 \leqslant i, j \leqslant p + 1 \Big)$$
$$\text{or } \Big(\bigvee_{i=1}^{n+1} c_i^p \leqslant \neg a \text{ and } c_i^p \wedge c_j^p = 0 \text{ for all distinct } i, j \text{ with } p + 2 \leqslant i, j \leqslant n + 1 \Big).$$

The right hand side of the first line of the above display implies that the sets of atoms below each of the c_i^p for $1 \leqslant i \leqslant p+1$ must be pairwise disjoint. Moreover, observe that $\boldsymbol{A}_n$ is finite and, therefore, each nonzero element is above an atom. Together with (15), this implies that there is at least one atom below each c_i^p. Consequently, there must be at least $p + 1$ distinct atoms below the join of $c_1^p, \dots, c_{p+1}^p$. Together with the left hand side of the first line of the above display, this implies that the number of atoms below a is $\geqslant p + 1$, which is false by assumption. Therefore,

$$\bigvee_{i=1}^{n+1} c_i^p \leqslant \neg a \text{ and } c_i^p \wedge c_j^p = 0 \text{ for all distinct } i, j \text{ with } p + 2 \leqslant i, j \leqslant n + 1.$$

As before, the right hand side of the above display and (15) imply that the number of distinct atoms below the join of $c_{p+2}^p, \dots, c_{n+1}^p$ must be at least $n - p$. Observe that by the left hand side of the above display and (6) it follows that every atom below the join of $c_{p+2}^p, \dots, c_{n+1}^p$ must be also below $\neg a$. As by assumption the number of atoms below $\neg a$ is precisely $n-p$, the set of atoms below $\neg a$ must coincide with the set of atoms below $c_{p+2}^p \vee \dots \vee c_{n+1}^p$. Therefore, using (6) we obtain

$$\neg a = \bigvee_{i=1}^{n+1} c_i^p.$$

Together with (13), this yields

$$a \vee \neg a \vee \neg\neg(a \vee \neg a) = a \vee \neg a \vee \neg\neg(a \vee \bigvee_{i=1}^{n+1} c_i^p) = d_p.$$

As $0 < a < e$ by assumption, from (3) and (4) it follows that $\neg\neg(a \vee \neg a) = \neg\neg e = 1$. Therefore, the above display yields

$$1 = a \vee \neg a \vee 1 = a \vee \neg a \vee \neg\neg(a \vee \neg a) = d_p.$$

By (11) we conclude that $b = 1$, as desired. It only remains to consider the case where the number of atoms below a is $\geqslant k + 1$. We need to prove that $b = e$. As $\boldsymbol{A}_n$ has n atoms by definition, the number of atoms below $\neg a$ is $\leqslant n - k - 1$ by (7). Then consider a positive $m \leqslant k$. Since $n - k - 1 < n - m$, the number of atoms below $\neg a$ is $< n - m$. Since (15) and the second line of (14) would imply that the number of atoms below $\neg a$ is $\geqslant n - m$, we conclude that the first line of (14) holds. Consequently,

$$\bigvee_{i=1}^{n+1} c_i^m \leqslant a. \tag{16}$$

We will prove that the following holds:

$$e = a \vee \neg a \leqslant a \vee \neg a \vee \neg\neg\Big(a \vee \bigvee_{i=1}^{n+1} c_i^m\Big) \leqslant a \vee \neg a \vee \neg\neg(a \vee a) = a \vee \neg a = e.$$

The first and the last steps above hold by $0 < a \leqslant e$ and (3), the second is straightforward, the third by (16) and (10), and the fourth by $a = \neg\neg a$, which follows from $a \neq e$ and (5). Together with (13), the above display yields $d_m = e$. As this holds for every $m \leqslant k$, from (11) it follows that $b = e$, as desired.

Next we prove the implication from right to left in the statement. Recall from the definition of $\varphi_{k,n}$ that it suffices to find c_i^m, d_m for $i \leqslant n+1$ and $m \leqslant k$ such that

$$\boldsymbol{A}_n \vDash \Big(b \approx \bigvee_{m=1}^{k} d_m\Big) \sqcap \prod_{m=1}^{k} \psi_{m,n}(a, d_m, c_1^m, \dots, c_{n+1}^m). \tag{17}$$

First, suppose that $a \in \{0, 1\}$. In this case, $b = 1$ by assumption. Choose $c_i^m = 0$ and $d_m = 1$ for all $i \leqslant n+1$ and $m \leqslant k$. Clearly, we have

$$b = 1 = \bigvee_{m=1}^{k} d_m.$$

From (2) it follows that for each $m \leqslant k$ we have $d(a) = 1$ and, therefore,

$$d(a) = 1 = d(0) = d(c_i^m) \text{ for each } i \leqslant n+1 \text{ and}$$

$$d(a) \vee \neg\neg\Big(a \vee \bigvee_{i=1}^{n+1} c_i^m\Big) = 1 \vee \neg\neg a = 1 = d_m,$$

which proves the validity of the first two conjuncts of $\psi_{m,n}$. Moreover, it holds that

$$\Big(\Big(\bigvee_{i=1}^{n+1} c_i^m \to a\Big) \wedge \bigwedge_{\substack{i,j=1\\ i\neq j}}^{m+1} \neg(c_i^m \wedge c_j^m)\Big) \vee \Big(\Big(\bigvee_{i=1}^{n+1} c_i^m \to \neg a\Big) \wedge \bigwedge_{\substack{i,j=m+2\\ i\neq j}}^{n+1} \neg(c_i^m \wedge c_j^m)\Big)$$

$$= \Big((0 \to a) \wedge \bigwedge_{\substack{i,j=1\\ i\neq j}}^{m+1} \neg 0\Big) \vee \Big((0 \to \neg a) \wedge \bigwedge_{\substack{i,j=m+2\\ i\neq j}}^{n+1} \neg 0\Big) = 1.$$

This establishes (17) for the case where $a \in \{0, 1\}$.

It only remains to consider the case where $0 < a < 1$. Observe that choosing $c_i^m \in \mathsf{at}(\boldsymbol{A}_n)$ for all $i \leqslant n+1$ and $m \leqslant k$ guarantees that

$$d(a) = e = d(c_i^m) \text{ for all } i \leqslant n+1 \text{ and } m \leqslant k \tag{18}$$

by (3). Moreover, (6) implies that, in order to guarantee that

$$\Big(\Big(\bigvee_{i=1}^{n+1} c_i^m \to a\Big) \wedge \bigwedge_{\substack{i,j=1\\ i\neq j}}^{m+1} \neg(c_i^m \wedge c_j^m)\Big) \vee \Big(\Big(\bigvee_{i=1}^{n+1} c_i^m \to \neg a\Big) \wedge \bigwedge_{\substack{i,j=m+2\\ i\neq j}}^{n+1} \neg(c_i^m \wedge c_j^m)\Big) = 1,$$

it suffices to choose c_i^m so that one of the following holds:

$$\{c_1^m, \ldots, c_{n+1}^m\} = \mathsf{at}_{\boldsymbol{A}_n}(a) \text{ and } c_i^m \neq c_j^m \text{ for all } i, j \in \{1, \ldots, m+1\} \text{ with } i \neq j, \tag{19}$$

$$\{c_1^m, \ldots, c_{n+1}^m\} = \mathsf{at}_{\boldsymbol{A}_n}(\neg a) \text{ and } c_i^m \neq c_j^m \text{ for all } i, j \in \{m+2, \ldots, n+1\} \text{ with } i \neq j. \tag{20}$$

We distinguish three cases. First, let $a = e$. Then $b = 1$ by assumption. Choose $c_i^m \in \mathsf{at}(\boldsymbol{A}_n) = \mathsf{at}_{\boldsymbol{A}_n}(e)$ for all $i \leqslant n+1$ and $m \leqslant k$ such that $\{c_1^m, \ldots, c_n^m\}$ are precisely the n distinct atoms of $\boldsymbol{A}_n$ and let $d_m = 1$ for each $m \leqslant k$. Then condition (19) is satisfied, since $m \leqslant k \leqslant n-1$, and thus $m+1 \leqslant n$. Therefore, to verify (17), it only remains to observe that for each $m \leqslant k$ we have

$$d(a) \vee \neg\neg\Big(a \vee \bigvee_{i=1}^{n+1} c_i^m\Big) = d(e) \vee \neg\neg(a \vee e) = d(e) \vee \neg\neg e = 1 = d_m,$$

which is true by (4) and $a = e$.

Next we consider the case where $0 < a < e$ and $|\mathsf{at}_{\boldsymbol{A}_n}(a)| = p \leqslant k$. Then $b = 1$ by assumption. For all $m < p$ and $i \leqslant n+1$ consider $c_i^m \in \mathsf{at}_{\boldsymbol{A}_n}(a)$ such that $\{c_1^m, \ldots, c_p^m\} = \mathsf{at}_{\boldsymbol{A}_n}(a)$ and $d_m = e$. Moreover, for all $p \leqslant m \leqslant k$ and $i \leqslant n+1$ consider $c_i^m \in \mathsf{at}_{\boldsymbol{A}_n}(\neg a)$ such that $\{c_{p+2}^m, \ldots, c_{n+1}^m\} = \mathsf{at}_{\boldsymbol{A}_n}(\neg a)$ and $d_m = 1$. Then for $m < p$ condition (19) is satisfied and by (3), (4), (5), and $0 < a < e$ we have

$$d(a) \vee \neg\neg\Big(a \vee \bigvee_{i=1}^{n+1} c_i^m\Big) = d(a) \vee \neg\neg(a \vee a) = e = d_m.$$

On the other hand, for every m such that $p \leqslant m \leqslant k$ condition (20) is satisfied. Moreover, using (3), (4), and $0 < a < e$, we obtain

$$d(a) \vee \neg\neg\Big(a \vee \bigvee_{i=1}^{n+1} c_i^m\Big) = d(a) \vee \neg\neg(a \vee \neg a) = e \vee \neg\neg e = 1 = d_m.$$

Since $1 = \bigvee_{m \leqslant k} d_m$ (because $d_k = 1$), this verifies that (17) holds.

It only remains to consider the case where $0 < a < e$ and $|\mathsf{at}_{\boldsymbol{A}_n}(a)| = p \geqslant k+1$. In this case, we have $b = e$ by assumption. Then for all $i \leqslant n+1$ and $m \leqslant k$ consider $c_i^m \in \mathsf{at}_{\boldsymbol{A}_n}(a)$ such that $\{c_1^m, \ldots, c_p^m\} = \mathsf{at}_{\boldsymbol{A}_n}(a)$. Also choose $d_m = e$ for each $m \leqslant k$. Then (19) it satisfied because $m+1 \leqslant k+1 \leqslant p$. Therefore, to conclude the proof of (17), it only remains to observe that for each $m \leqslant k$ we have

$$d(a) \vee \neg\neg\Big(a \vee \bigvee_{i=1}^{n+1} c_i^m\Big) = d(a) \vee \neg\neg(a \vee a) = e = d_m,$$

which holds by (3), (5), and $0 < a < e$. This completes the proof. ⊠

As a consequence of Proposition 3.7, we get the following.

Corollary 3.8. *For every $n \geqslant 3$ and positive $k \leqslant n-1$ the formula $\varphi_{k,n}$ defines an implicit operation $f_{k,n} \in \mathsf{ext}_{pp}(\mathbb{V}(\boldsymbol{A}_n))$ such that $f_{k,n}^{\boldsymbol{A}_n}$ is total and for every $a \in A_n$,*

$$f_{k,n}^{\boldsymbol{A}_n}(a) = \begin{cases} 1 & \text{if } a \in \{0, e, 1\}; \\ 1 & \text{if } 0 < a < e \text{ and } |\mathsf{at}_{\boldsymbol{A}_n}(a)| \leqslant k; \\ e & \text{if } 0 < a < e \text{ and } |\mathsf{at}_{\boldsymbol{A}_n}(a)| \geqslant k+1. \end{cases}$$

Proof. In view of Proposition 3.7, the pp formula $\varphi_{k,n}$ is functional in $\boldsymbol{A}_n$. By [4, Cor. 3.11] it is also functional in $\mathbb{Q}(\boldsymbol{A}_n)$. In view of Corollary 3.6, this means that $\varphi_{k,n}$ is functional in $\mathbb{V}(\boldsymbol{A}_n)$ and, therefore, defines an implicit operation $f_{k,n} \in \mathsf{imp}_{pp}(\mathbb{V}(\boldsymbol{A}_n))$. From Proposition 3.7 it follows that $f_{k,n}^{\boldsymbol{A}_n}$ is total and defined as in the statement. As $f_{k,n}^{\boldsymbol{A}_n}$ is total, we can apply [4, Prop. 8.11(ii)] to the case where $\mathsf{K} = \mathbb{V}(\boldsymbol{A}_n) = \mathbb{Q}(\boldsymbol{A}_n)$ and $\mathsf{M} = \{\boldsymbol{A}_n\}$, obtaining that $f_{k,n}$ is extendable. Thus, we conclude that $f_{k,n} \in \mathsf{ext}_{pp}(\mathbb{V}(\boldsymbol{A}_n))$. ⊠

Now, for every $n \geqslant 3$ let

$$\mathcal{F}_n = \{f_{k,n} : k \text{ is positive and } \leqslant n-1\}.$$

Observe that $\mathcal{F}_n \subseteq \mathsf{ext}_{pp}(\mathbb{V}(\boldsymbol{A}_n))$ by Corollary 3.8. Then consider an $\mathcal{F}_n$-expansion

$$\mathscr{L}_{\mathcal{F}_n} = \mathscr{L} \cup \{\ell_f : f \in \mathcal{F}_n\}$$

of the language $\mathscr{L}$ of Heyting algebras and let

$$\mathsf{B}(n) = \mathbb{S}(\mathbb{V}(\boldsymbol{A}_n)[\mathscr{L}_{\mathcal{F}_n}])$$

be the corresponding pp expansion of $\mathbb{V}(\boldsymbol{A}_n)$. Our aim is to prove the following.

Theorem 3.9. *Let $n \geqslant 3$. Then $\mathsf{B}(n)$ is a congruence preserving Beth companion of $\mathbb{V}(\boldsymbol{A}_n)$.*

To this end, recall from Corollary 3.8 that $f^{\boldsymbol{A}_n}$ is total for every $f \in \mathcal{F}_n$, whence the algebra

$$\boldsymbol{B}_n = \boldsymbol{A}_n[\mathscr{L}_{\mathcal{F}_n}]$$

is defined. We begin with the following observation.

Proposition 3.10. *For every $n \geqslant 3$ we have*

$$\mathsf{B}(n) = \mathbb{V}(\boldsymbol{B}_n) \quad \textit{and} \quad \mathsf{B}(n)_{\text{FSI}} = \mathbb{IS}(\boldsymbol{B}_n).$$

Moreover, $\mathsf{B}(n)$ is an arithmetical variety.

Proof. We begin with the following observation.

Claim 3.11. *We have $\mathbb{V}(\boldsymbol{B}_n)_{\text{FSI}} = \mathbb{IS}(\boldsymbol{B}_n)$.*

Proof of the Claim. First, we show that

$$\mathsf{Con}(\boldsymbol{C}) = \mathsf{Con}(\boldsymbol{C}{\restriction}_{\mathscr{L}}) \text{ for every } \boldsymbol{C} \in \mathbb{IS}(\boldsymbol{B}_n). \tag{21}$$

Clearly, it will be enough to prove the above display for an arbitrary $\boldsymbol{C} \in \mathbb{S}(\boldsymbol{B}_n)$. The inclusion $\mathsf{Con}(\boldsymbol{C}) \subseteq \mathsf{Con}(\boldsymbol{C}{\restriction}_{\mathscr{L}})$ is straightforward. To prove the reverse one, consider $\theta \in \mathsf{Con}(\boldsymbol{C}{\restriction}_{\mathscr{L}})$. From $\boldsymbol{C} \leqslant \boldsymbol{B}_n$ it follows that $\boldsymbol{C}{\restriction}_{\mathscr{L}} \leqslant (\boldsymbol{B}_n){\restriction}_{\mathscr{L}} = \boldsymbol{A}_n$. As $\boldsymbol{C}{\restriction}_{\mathscr{L}}$ and $\boldsymbol{A}_n$ are Heyting algebras and the variety of Heyting algebras has the congruence extension property, there exists

$\phi \in \mathsf{Con}(\boldsymbol{A}_n)$ such that $\theta = \phi{\upharpoonright}_C$. From [4, Prop. 12.13] and the definition of $\boldsymbol{B}_n$ it follows that $\mathsf{Con}(\boldsymbol{A}_n) = \mathsf{Con}(\boldsymbol{B}_n)$. Therefore, $\phi \in \mathsf{Con}(\boldsymbol{B}_n)$. Together with $\boldsymbol{C} \leqslant \boldsymbol{B}_n$, this yields $\theta = \phi{\upharpoonright}_C \in \mathsf{Con}(\boldsymbol{C})$, as desired.

Next, we prove $\mathbb{V}(\boldsymbol{B}_n)_{\text{FSI}} = \mathbb{IS}(\boldsymbol{B}_n)$. By Proposition 3.4 we have $\mathbb{V}(\boldsymbol{B}_n)_{\text{FSI}} \subseteq \mathbb{HS}(\boldsymbol{B}_n)$. Therefore, it suffices to show that the finitely subdirectly irreducible members of $\mathbb{HS}(\boldsymbol{B}_n)$ are precisely the members of $\mathbb{IS}(\boldsymbol{B}_n)$. To this end, consider a finitely subdirectly irreducible $\boldsymbol{C} \in \mathbb{HS}(\boldsymbol{B}_n)$. Then there exist $\boldsymbol{D} \leqslant \boldsymbol{B}_n$ and $\theta \in \mathsf{Con}(\boldsymbol{D})$ such that $\boldsymbol{C} \cong \boldsymbol{D}/\theta$. By [4, Prop. 2.10] the congruence θ is meet irreducible in $\mathsf{Con}(\boldsymbol{D})$. By (21) it is also a meet irreducible member of $\mathsf{Con}(\boldsymbol{D}{\upharpoonright}_{\mathscr{L}})$. Since $\boldsymbol{D}{\upharpoonright}_{\mathscr{L}} \leqslant \boldsymbol{A}_n$, one can check by inspection that the only meet irreducible congruences of $\boldsymbol{D}{\upharpoonright}_{\mathscr{L}}$ are id_D and the congruences ϕ of $\boldsymbol{D}{\upharpoonright}_{\mathscr{L}}$ with exactly two equivalences, namely, $0/\phi$ and $1/\phi$. If $\theta = \mathrm{id}_D$, then $\boldsymbol{C} \cong \boldsymbol{D}$ and, therefore, $\boldsymbol{C} \in \mathbb{IS}(\boldsymbol{B}_n)$ because $\boldsymbol{D} \leqslant \boldsymbol{B}_n$. On the other hand, if θ has exactly two equivalence classes $0/\theta$ and $1/\theta$, then $\boldsymbol{D}/\theta$ is isomorphic to the subalgebra of $\boldsymbol{B}_n$ with universe $\{0, 1\}$, whence $\boldsymbol{C} \in \mathbb{IS}(\boldsymbol{B}_n)$. Finally, we show that every member of $\mathbb{IS}(\boldsymbol{B}_n)$ is finitely subdirectly irreducible. Let $\boldsymbol{C} \in \mathbb{IS}(\boldsymbol{B}_n)$. Then $\mathsf{Con}(\boldsymbol{C}) = \mathsf{Con}(\boldsymbol{C}{\upharpoonright}_{\mathscr{L}})$ by (21). Since $\boldsymbol{C} \in \mathbb{IS}(\boldsymbol{B}_n)$, the definition of $\boldsymbol{B}_n$ guarantees that $\boldsymbol{C}{\upharpoonright}_{\mathscr{L}} \in \mathbb{IS}(\boldsymbol{A}_n)$. By inspection one can check that every member of $\mathbb{IS}(\boldsymbol{A}_n)$ is finitely subdirectly irreducible. Consequently, so is $\boldsymbol{C}{\upharpoonright}_{\mathscr{L}}$. By [4, Prop. 2.10] the congruence id_C is meet irreducible in $\mathsf{Con}(\boldsymbol{C}{\upharpoonright}_{\mathscr{L}})$. As $\mathsf{Con}(\boldsymbol{C}) = \mathsf{Con}(\boldsymbol{C}{\upharpoonright}_{\mathscr{L}})$, it is also meet irreducible in $\mathsf{Con}(\boldsymbol{C})$. Hence, we conclude that $\boldsymbol{C}$ is finitely subdirectly irreducible by [4, Prop. 2.10]. ⊠

By Claim 3.11 and the Subdirect Decomposition Theorem (see, e.g., [7, Thm. 3.1.1]) we obtain $\mathbb{V}(\boldsymbol{B}_n) = \mathbb{ISP}(\mathbb{V}(\boldsymbol{B}_n)_{\text{FSI}}) = \mathbb{ISPIS}(\boldsymbol{B}_n)$. Consequently, $\mathbb{V}(\boldsymbol{B}_n) \subseteq \mathbb{Q}(\boldsymbol{B}_n)$. As the reverse inclusion always holds, we conclude that $\mathbb{V}(\boldsymbol{B}_n) = \mathbb{Q}(\boldsymbol{B}_n)$.

Now, recall from Corollary 3.6 that $\mathbb{V}(\boldsymbol{A}_n) = \mathbb{Q}(\boldsymbol{A}_n)$. As $\boldsymbol{B}_n = \boldsymbol{A}_n[\mathscr{L}_{\mathcal{F}_n}]$, this allows us to apply [4, Thm. 10.5] to the case where $\mathsf{K} = \mathbb{V}(\boldsymbol{A}_n)$, $\mathsf{N} = \{\boldsymbol{A}_n\}$, and $\mathbb{O} = \mathbb{Q}$, obtaining $\mathsf{B}(n) = \mathbb{S}(\mathbb{V}(\boldsymbol{A}_n)[\mathscr{L}_{\mathcal{F}_n}]) = \mathbb{Q}(\boldsymbol{A}_n[\mathscr{L}_{\mathcal{F}_n}]) = \mathbb{Q}(\boldsymbol{B}_n)$. Since $\mathbb{Q}(\boldsymbol{B}_n) = \mathbb{V}(\boldsymbol{B}_n)$, we obtain $\mathsf{B}(n) = \mathbb{V}(\boldsymbol{B}_n)$. Therefore, $\mathsf{B}(n)_{\text{FSI}} = \mathbb{V}(\boldsymbol{B}_n)_{\text{FSI}} = \mathbb{IS}(\boldsymbol{B}_n)$. Lastly, since $\boldsymbol{B}_n$ has a Heyting algebra reduct, the variety $\mathbb{V}(\boldsymbol{B}_n)$ is arithmetical (see, e.g., [3, p. 80]). ⊠

An *endomorphism* of an algebra $\boldsymbol{A}$ is a homomorphism $h\colon \boldsymbol{A} \to \boldsymbol{A}$. When h is an isomorphism, we say that it is an *automorphism* of $\boldsymbol{A}$. The sets of endomorphisms and of automorphisms of $\boldsymbol{A}$ will be denoted, respectively, by $\mathsf{end}(\boldsymbol{A})$ and $\mathsf{aut}(\boldsymbol{A})$.

Similarly to the case of complete atomic Boolean algebras (cf. [6, Cor. 14.2]), one can easily verify that every permutation of the atoms of $\boldsymbol{A}_n$ for some $n \in \mathbb{N}$ induces an automorphism of $\boldsymbol{A}_n$ in the following way.

Proposition 3.12. *Let $n \in \mathbb{N}$ and let $\sigma\colon \mathsf{at}(\boldsymbol{A}_n) \to \mathsf{at}(\boldsymbol{A}_n)$ be a permutation. Then the map $\sigma^*\colon \boldsymbol{A}_n \to \boldsymbol{A}_n$ defined for every $a \in \boldsymbol{A}_n$ as*

$$\sigma^*(a) = \begin{cases} 1 & \text{if } a = 1; \\ \bigvee \sigma[\mathsf{at}_{\boldsymbol{A}_n}(a)] & \text{if } a \neq 1 \end{cases}$$

is an automorphism of $\boldsymbol{A}_n$.

We will also make use of the next observation on the automorphisms of $\boldsymbol{B}_n$.

Proposition 3.13. *The following conditions hold for every $n \geqslant 3$:*

(i) *for every $\boldsymbol{A} \leqslant \boldsymbol{B}_n$ and $b \in B_n - (A \cup \{e\})$ there exists $h \in \mathsf{aut}(\boldsymbol{B}_n)$ such that $b \neq h(b)$ and $a = h(a)$ for every $a \in A$;*

(ii) *for every pair of embeddings $g, h \colon \boldsymbol{A} \to \boldsymbol{B}_n$ there exists $i \in \mathsf{aut}(\boldsymbol{B}_n)$ such that $g = i \circ h$.*

Proof. (i): Consider $\boldsymbol{A} \leqslant \boldsymbol{B}_n$ and $b \in B_n - (A \cup \{e\})$. For every $a \in \mathsf{at}(\boldsymbol{A})$ let

$$X_a = \mathsf{at}_{\boldsymbol{B}_n}(a).$$

We will prove that $\{X_a : a \in \mathsf{at}(\boldsymbol{A})\}$ forms a partition of $\mathsf{at}(\boldsymbol{B}_n)$. As $\boldsymbol{A} \leqslant \boldsymbol{B}_n$, for every distinct $a, c \in \mathsf{at}(\boldsymbol{A})$ we have $X_a \cap X_c = \emptyset$. Therefore, it only remains to show that for every $a \in \mathsf{at}(\boldsymbol{B}_n)$ there exists $c \in \mathsf{at}(\boldsymbol{A})$ such that $a \in X_c$, i.e., $a \leqslant c$. Consider $a \in \mathsf{at}(\boldsymbol{B}_n)$. We begin by showing that $e \leqslant \bigvee \mathsf{at}(\boldsymbol{A})$. If $A = \{0, 1\}$, we have $1 \in \mathsf{at}(\boldsymbol{A})$ and, therefore, $e \leqslant \bigvee \mathsf{at}(\boldsymbol{A}) = 1$. Then we consider the case where $A \neq \{0, 1\}$. In this case, there exists $a \in A$ such that $0 < a < 1$. Observe that $\neg a \in A$ and $\mathsf{at}_{\boldsymbol{A}}(a) \cup \mathsf{at}_{\boldsymbol{A}}(\neg a) \subseteq \mathsf{at}(\boldsymbol{A})$. Consequently, using (3) and (6), we obtain

$$e = a \vee \neg a = \bigvee \mathsf{at}_{\boldsymbol{A}}(a) \vee \bigvee \mathsf{at}_{\boldsymbol{A}}(\neg a) \leqslant \bigvee \mathsf{at}(\boldsymbol{A}).$$

Hence, we conclude that $e \leqslant \bigvee \mathsf{at}(\boldsymbol{A})$, as desired. Therefore, $a \leqslant \bigvee \mathsf{at}(\boldsymbol{A})$ because $a \in \mathsf{at}(\boldsymbol{B}_n)$ and every atom of $\boldsymbol{B}_n$ is below e. Since $a \in \mathsf{at}(\boldsymbol{B}_n)$, from $a \leqslant \bigvee \mathsf{at}(\boldsymbol{A})$ it follows that $a \leqslant c$ for some $c \in \mathsf{at}(\boldsymbol{A})$. Hence, $\{X_a : a \in \mathsf{at}(\boldsymbol{A})\}$ forms a partition of $\mathsf{at}(\boldsymbol{B}_n)$, as desired.

Now, observe that $1 \in A$ because $\boldsymbol{A} \leqslant \boldsymbol{B}_n$. Together with the assumption that $b \notin A \cup \{e\}$, this yields $b < e$. We will show that there exist $a \in \mathsf{at}(\boldsymbol{A})$ and $c, d \in X_a$ such that $c \leqslant b$ and $d \nleqslant b$. We have two cases: either $A = \{0, 1\}$ or $A \neq \{0, 1\}$. First, suppose that $A = \{0, 1\}$. Then $\mathsf{at}(\boldsymbol{A}) = \{1\}$ and $X_1 = \mathsf{at}(\boldsymbol{B}_n)$. Since $b < e$, there exist $c, d \in \mathsf{at}(\boldsymbol{B}_n) = X_1$ such that $c \leqslant b$ and $d \nleqslant b$, as desired. Next we consider the case where $A \neq \{0, 1\}$. Recall from the first part of the proof that $\{X_a : a \in \mathsf{at}(\boldsymbol{A})\}$ is a partition of $\mathsf{at}(\boldsymbol{B}_n)$. Therefore, $\mathsf{at}_{\boldsymbol{B}_n}(b) \subseteq \mathsf{at}(\boldsymbol{B}_n) = \bigcup \{X_a : a \in \mathsf{at}(\boldsymbol{A})\}$. Suppose, with a view to contradiction, that for every $a \in \mathsf{at}(\boldsymbol{A})$ we have $X_a \subseteq \mathsf{at}_{\boldsymbol{B}_n}(b)$ or $X_a \cap \mathsf{at}_{\boldsymbol{B}_n}(b) = \emptyset$. Then

$$\mathsf{at}_{\boldsymbol{B}_n}(b) = \bigcup \{X_a : a \in \mathsf{at}(\boldsymbol{A}) \text{ and } X_a \subseteq \mathsf{at}_{\boldsymbol{B}_n}(b)\}. \tag{22}$$

It follows that

$$\begin{aligned} b = \bigvee \mathsf{at}_{\boldsymbol{B}_n}(b) &= \bigvee \bigcup \{X_a : a \in \mathsf{at}(\boldsymbol{A}) \text{ and } X_a \subseteq \mathsf{at}_{\boldsymbol{B}_n}(b)\} \\ &= \bigvee \left\{ \bigvee \mathsf{at}_{\boldsymbol{B}_n}(a) : a \in \mathsf{at}(\boldsymbol{A}) \text{ and } X_a \subseteq \mathsf{at}_{\boldsymbol{B}_n}(b) \right\} \\ &= \bigvee \{a \in \mathsf{at}(\boldsymbol{A}) : X_a \subseteq \mathsf{at}_{\boldsymbol{B}_n}(b)\}, \end{aligned}$$

where the first equality holds by (6) and $b \neq 1$ (the latter follows from $b \notin A$), the second by (22), the third by the definition of X_a, and the fourth follows from (6) because $a \neq 1$ (the latter holds because $a \in \mathsf{at}(\boldsymbol{A})$ and $A \neq \{0, 1\}$). But this is a contradiction to the assumption that $b \notin A$. Therefore, there exists $a \in \mathsf{at}(\boldsymbol{A})$ such that $\emptyset \subsetneq X_a \cap \mathsf{at}_{\boldsymbol{B}_n}(b) \subsetneq X_a$. Consequently, we can choose $c \in X_a \cap \mathsf{at}_{\boldsymbol{B}_n}(b)$ to obtain $c \in X_a$ such that $c \leqslant b$ and $d \in X_a - \mathsf{at}_{\boldsymbol{B}_n}(b)$ such that $d \nleqslant b$. Thus, in both cases, there exist $a \in \mathsf{at}(\boldsymbol{A})$ and $c, d \in B_n$ with

$$c, d \in X_a, \qquad c \in \mathsf{at}_{\boldsymbol{B}_n}(b), \quad \text{and} \quad d \nleqslant b. \tag{23}$$

Then let $\sigma\colon \mathsf{at}(\boldsymbol{B}_n) \to \mathsf{at}(\boldsymbol{B}_n)$ be a permutation such that

$$\sigma[X_a] = X_a \text{ for every } a \in \mathsf{at}(\boldsymbol{A}) \text{ and } \sigma(c) = d. \tag{24}$$

Notice that σ exists because $c, d \in X_a$ by the first item of (23). Recall that $\boldsymbol{B}_n = \boldsymbol{A}_n[\mathscr{L}_{\mathcal{F}_n}]$. Thus, we can consider the automorphism $\sigma^*\colon \boldsymbol{A}_n \to \boldsymbol{A}_n$ defined in Proposition 3.12, which by [4, Prop. 9.5] is also an automorphism of $\boldsymbol{B}_n$. Therefore, in order to complete the proof, it only remains to show that $\sigma^*(b) \neq b$ and $\sigma^*(a) = a$ for every $a \in A$.

We begin by proving that

$$\sigma^*(b) = \bigvee \sigma[\mathsf{at}_{\boldsymbol{B}_n}(b)] \geqslant \sigma(c) = d.$$

The first step in the above display holds by the definition of σ^* and $b < e < 1$, the second by the second item of (23), and the third by the right hand side of (24). Together with the third item of (23), the above display yields $\sigma^*(b) \neq b$.

Lastly, we will prove that $\sigma^*(a) = a$ for every $a \in A$. Consider $a \in A$. If $a = 1$, then $\sigma^*(a) = a$ by the definition of σ^*. Then we consider the case where $a \neq 1$. We will prove that

$$\begin{aligned}\sigma^*(a) &= \sigma^*\Big(\bigvee^{\boldsymbol{A}} \mathsf{at}_{\boldsymbol{A}}(a)\Big) = \sigma^*\Big(\bigvee^{\boldsymbol{B}_n} \mathsf{at}_{\boldsymbol{A}}(a)\Big) = \bigvee^{\boldsymbol{B}_n} \sigma^*[\mathsf{at}_{\boldsymbol{A}}(a)] = \bigvee^{\boldsymbol{B}_n}_{p \in \mathsf{at}_{\boldsymbol{A}}(a)} \Big(\bigvee^{\boldsymbol{B}_n} \sigma[\mathsf{at}_{\boldsymbol{B}_n}(p)]\Big) \\ &= \bigvee^{\boldsymbol{B}_n}_{p \in \mathsf{at}_{\boldsymbol{A}}(a)} \Big(\bigvee^{\boldsymbol{B}_n} \mathsf{at}_{\boldsymbol{B}_n}(p)\Big) = \bigvee^{\boldsymbol{B}_n} \mathsf{at}_{\boldsymbol{A}}(a) = \bigvee^{\boldsymbol{A}} \mathsf{at}_{\boldsymbol{A}}(a) = a.\end{aligned}$$

The above equalities are justified as follows: the first and the last hold by (6) and $a \neq 1$, the second and the second to last because $\boldsymbol{A} \leqslant \boldsymbol{B}_n$, the third because σ^* is a homomorphism of bounded lattices and, therefore, it preserves finite (possibly empty) joins, the fourth by the definition of σ^* and the fact that $p \leqslant a < 1$ implies $p \neq 1$, the fifth by the left hand side of (24), and the sixth because $p \leqslant a < 1$ implies $p \leqslant e$, whence (6) guarantees that $p = \bigvee^{\boldsymbol{B}_n} \mathsf{at}_{\boldsymbol{B}_n}(p)$. Thus, we conclude that $\sigma^*(a) = a$ for every $a \in A$.

(ii): Consider a pair of embeddings $g, h\colon \boldsymbol{A} \to \boldsymbol{B}_n$. As g and h are homomorphisms of bounded lattices, we have $g(0) = h(0) = 0$ and $g(1) = h(1) = 1$. Therefore, if $A = \{0, 1\}$, we have $g = h$ and we are done letting i be the identity map on B_n.

Then we may assume that $A \neq \{0, 1\}$, that is, $\{0, 1\} \subsetneq A$. Since $g, h\colon \boldsymbol{A} \to \boldsymbol{B}_n$ are embeddings, both $g[\boldsymbol{A}]$ and $h[\boldsymbol{A}]$ are subalgebras of $\boldsymbol{B}_n$ containing at least an element a other than 0 and 1. Then they must also contain $\neg a$ and, therefore, $e = a \vee \neg a \in g[\boldsymbol{A}] \cap h[\boldsymbol{A}]$ by (3). As e is the second largest element of $\boldsymbol{B}_n$ and g and h are embeddings of lattices, we obtain that $\boldsymbol{A}$ possesses a second largest element e^* such that $g(e^*) = h(e^*) = e$. Moreover, $0 < e^* < 1$ because e^* is the second largest element to $\boldsymbol{A}$ and $A \neq \{0, 1\}$. If $A = \{0, e^*, 1\}$, we have $g = h$ and we are done letting i be the identity map on B_n.

Then we may assume that $A \neq \{0, e^*, 1\}$, that is, $\{0, e^*, 1\} \subsetneq A$. We rely on the following series of observations.

Claim 3.14. *We have* $g[\mathsf{at}(\boldsymbol{A})] \cup h[\mathsf{at}(\boldsymbol{A})] \subseteq \{a \in B_n : 0 < a < e\}$.

Proof of the Claim. By symmetry it suffices to show that $g[\mathsf{at}(\boldsymbol{A})] \subseteq \{a \in B_n : 0 < a < e\}$. To this end, consider $a \in \mathsf{at}(\boldsymbol{A})$. Then $a > 0$. Moreover, since e^* is the second largest element of $\boldsymbol{A}$ and $\boldsymbol{A}$ contains an element other than $0, e^*$, and 1, from $a \in \mathsf{at}(\boldsymbol{A})$ it follows that $a < e^*$. Therefore, $0 < a < e^*$. Since g is a embedding of bounded lattices, we obtain $0 = g(0) < g(a) < g(e^*)$. As we already established $g(e^*) = e$, we conclude that $0 < g(a) < e$. ⊠

Claim 3.15. *For every $a \in \mathsf{at}(\boldsymbol{A})$ we have $|\mathsf{at}_{\boldsymbol{B}_n}(g(a))| = |\mathsf{at}_{\boldsymbol{B}_n}(h(a))|$.*

Proof of the Claim. Recall that $\boldsymbol{A}_n$ has n atoms by definition. As $\boldsymbol{B}_n$ is an expansion of $\boldsymbol{A}_n$, we obtain that $\boldsymbol{B}_n$ has n atoms as well. Then consider $a \in B_n - \{0, e, 1\}$ and observe that $|\mathsf{at}_{\boldsymbol{B}_n}(a)| \leqslant n - 1$ because $|\mathsf{at}_{\boldsymbol{B}_n}(a)| = n$ by (6) would imply $a \geqslant e$. Recall that $\mathscr{L}_{\mathcal{F}_n} = \mathscr{L} \cup \{\ell_f : f \in \mathcal{F}_n\}$. Therefore, from Corollary 3.8 and $\ell_{f_{k,n}}^{\boldsymbol{B}_n} = f_{k,n}^{\boldsymbol{A}_n}$ it follows that for every $m \leqslant n - 1$,

$$|\mathsf{at}_{\boldsymbol{B}_n}(a)| = m \iff \text{for every } 0 < k \leqslant n-1 \text{ we have } \ell_{f_{k,n}}^{\boldsymbol{B}_n}(a) = \begin{cases} 1 & \text{if } m \leqslant k; \\ e & \text{if } m \geqslant k+1. \end{cases} \tag{25}$$

To prove the statement of the claim, consider $a \in \mathsf{at}(\boldsymbol{A})$. By Claim 3.14 we have $0 < g(a), h(a) < e$. Then $|\mathsf{at}_{\boldsymbol{B}_n}(g(a))|$ is a positive integer $m \leqslant n - 1$. In view of (25), for every positive $k \leqslant n - 1$,

$$\ell_{f_{k,n}}^{\boldsymbol{B}_n}(g(a)) = \begin{cases} 1 & \text{if } m \leqslant k; \\ e & \text{if } m \geqslant k+1. \end{cases}$$

Since $g\colon \boldsymbol{A} \to \boldsymbol{B}_n$ is an embedding such that $g(e^*) = e$ and $g(1) = 1$, this yields that for every positive $k \leqslant n - 1$,

$$\ell_{f_{k,n}}^{\boldsymbol{A}}(a) = \begin{cases} 1 & \text{if } m \leqslant k; \\ e^* & \text{if } m \geqslant k+1. \end{cases}$$

As $h\colon \boldsymbol{A} \to \boldsymbol{B}_n$ is also an embedding such that $h(e^*) = e$ and $h(1) = 1$, we obtain that for every positive $k \leqslant n - 1$,

$$\ell_{f_{k,n}}^{\boldsymbol{B}_n}(h(a)) = \begin{cases} 1 & \text{if } m \leqslant k; \\ e & \text{if } m \geqslant k+1. \end{cases}$$

Together with (25), this yields $|\mathsf{at}_{\boldsymbol{B}_n}(h(a))| = m$. ⊠

Claim 3.16. *For every $a, b \in \mathsf{at}(\boldsymbol{A})$,*

$$\textit{if } a \neq b, \textit{ then } \mathsf{at}_{\boldsymbol{B}_n}(g(a)) \cap \mathsf{at}_{\boldsymbol{B}_n}(g(b)) = \emptyset = \mathsf{at}_{\boldsymbol{B}_n}(h(a)) \cap \mathsf{at}_{\boldsymbol{B}_n}(h(b)).$$

Proof of the Claim. Suppose that $a \neq b$. By symmetry it suffices to show that $\mathsf{at}_{\boldsymbol{B}_n}(g(a)) \cap \mathsf{at}_{\boldsymbol{B}_n}(g(b)) = \emptyset$. From $a \neq b$ and $a, b \in \mathsf{at}(\boldsymbol{A})$ it follows that $a \wedge^{\boldsymbol{A}} b = 0$. Consequently, $g(a) \wedge^{\boldsymbol{B}_n} g(b) = 0$ because $g\colon \boldsymbol{A} \to \boldsymbol{B}_n$ is an embedding. Therefore, we conclude that $\mathsf{at}_{\boldsymbol{B}_n}(g(a)) \cap \mathsf{at}_{\boldsymbol{B}_n}(g(b)) = \emptyset$. ⊠

In view of Claims 3.15 and 3.16 there exists a permutation $\sigma\colon \mathsf{at}(\boldsymbol{B}_n) \to \mathsf{at}(\boldsymbol{B}_n)$ such that

$$\sigma[\mathsf{at}_{\boldsymbol{B}_n}(h(a))] = \mathsf{at}_{\boldsymbol{B}_n}(g(a)) \text{ for every } a \in \mathsf{at}(\boldsymbol{A}). \tag{26}$$

As $\boldsymbol{B}_n = \boldsymbol{A}_n[\mathscr{L}_{\mathcal{F}_n}]$, the map σ can also be viewed as a permutation of $\mathsf{at}(\boldsymbol{A}_n)$. Consequently, Proposition 3.12 yields an automorphism $\sigma^*\colon \boldsymbol{A}_n \to \boldsymbol{A}_n$, which by [4, Prop. 9.5] is also an automorphism of $\boldsymbol{B}_n$. To conclude the proof, it only remains to show that $g = \sigma^* \circ h$, for in this case we can take $i = \sigma^*$.

From the assumption that g, h, and σ^* are homomorphisms of bounded lattices it follows that $g(1) = h(1) = \sigma^*(1) = 1$, whence $g(1) = \sigma^*(h(1))$. Therefore, it suffices to show that $g(a) = \sigma^*(h(a))$ for every $a \in A - \{1\}$. We will prove that for every $a \in A - \{1\}$,

$$g(a) = g\Big(\bigvee^{\boldsymbol{A}} \mathsf{at}_{\boldsymbol{A}}(a)\Big) = \bigvee^{\boldsymbol{B}_n} g[\mathsf{at}_{\boldsymbol{A}}(a)] = \bigvee_{b \in \mathsf{at}_{\boldsymbol{A}}(a)}^{\boldsymbol{B}_n} \bigvee^{\boldsymbol{B}_n} \mathsf{at}_{\boldsymbol{B}_n}(g(b)) = \bigvee_{b \in \mathsf{at}_{\boldsymbol{A}}(a)}^{\boldsymbol{B}_n} \bigvee^{\boldsymbol{B}_n} \sigma[\mathsf{at}_{\boldsymbol{B}_n}(h(b))]$$

$$= \bigvee_{b \in \mathsf{at}_{\boldsymbol{A}}(a)}^{\boldsymbol{B}_n} \sigma^*(h(b)) = \sigma^*\Big(h\Big(\bigvee^{\boldsymbol{A}} \mathsf{at}_{\boldsymbol{A}}(a)\Big)\Big) = \sigma^*(h(a)).$$

The above equalities are justified as follows. The first and the last hold by (6) and the assumption that $a \neq 1$, the second and the second to last because g, h, and σ^* preserve finite (possibly empty) joins because they are homomorphisms of bounded lattices, the third by Claim 3.14 and (6), the fourth by (26), and the fifth follows from Claim 3.14 and the definition of σ^*. Hence, we conclude that $g = \sigma^* \circ h$. ⊠

Finalizing the proof of the fact that $\mathsf{B}(n)$ is a congruence preserving Beth companion of $\mathbb{V}(\boldsymbol{A}_n)$ (Theorem 3.9) requires some further investigation of the variety $\mathsf{B}(n)$ and its properties. While $\mathbb{V}(\boldsymbol{A}_n)$ lacks the amalgamation property for every $n \geqslant 3$ (see [10, Thm. 2]), this property holds in the pp expansion $\mathsf{B}(n)$ of $\mathbb{V}(\boldsymbol{A}_n)$, as we proceed to illustrate. To this end, we will employ the following result [5, Thm. 3.4][2] (see also [8, Thm. 3]), together with the observation that $\mathsf{B}(n)$ has the congruence extension property for each $n \geqslant 3$.

Given a quasivariety K, let

$$\mathsf{K}^*_{\text{RFSI}} = \mathsf{K}_{\text{RFSI}} \cup \{\boldsymbol{A} \in \mathsf{K} : \boldsymbol{A} \text{ is trivial}\}.$$

Theorem 3.17. *Let K be a quasivariety with the relative congruence extension property such that K_{RFSI} is closed under nontrivial subalgebras. Then K has the amalgamation property if and only if $\mathsf{K}^*_{\text{RFSI}}$ has the amalgamation property.*

To show that $\mathsf{B}(n)$ has the congruence extension property for each $n \geqslant 3$, we rely on the following preservation result.

Proposition 3.18. *Let M be a pp expansion of a quasivariety K. If K has the relative congruence extension property, then so does M.*

Proof. Suppose that K has the relative congruence extension property. Then consider $\boldsymbol{A} \leqslant \boldsymbol{B} \in \mathsf{M}$ and $\theta \in \mathsf{Con}_{\mathsf{M}}(\boldsymbol{A})$. We need to find some $\phi \in \mathsf{Con}_{\mathsf{M}}(\boldsymbol{B})$ such that $\theta = \phi{\upharpoonright}_A$. Since $\boldsymbol{A} \in \mathbb{S}(\mathsf{M}) = \mathsf{M}$, from [4, Rem. 12.2] it follows that $\mathsf{Con}_{\mathsf{M}}(\boldsymbol{A}) \subseteq \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}{\upharpoonright}_{\mathscr{L}_{\mathsf{K}}})$, whence $\theta \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}{\upharpoonright}_{\mathscr{L}_{\mathsf{K}}})$. Since M is a pp expansion of K, it is of the form $\mathbb{S}(\mathsf{K}[\mathscr{L}_{\mathcal{F}}])$. Together with $\boldsymbol{A} \leqslant \boldsymbol{B} \in \mathsf{M}$, this implies $\boldsymbol{A} \leqslant \boldsymbol{B} \leqslant \boldsymbol{C}$ for some $\boldsymbol{C} \in \mathsf{K}[\mathscr{L}_{\mathcal{F}}]$. Consequently, $\boldsymbol{A}{\upharpoonright}_{\mathscr{L}_{\mathsf{K}}} \leqslant \boldsymbol{C}{\upharpoonright}_{\mathscr{L}_{\mathsf{K}}} \in \mathsf{K}$. As $\theta \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}{\upharpoonright}_{\mathscr{L}_{\mathsf{K}}})$ and K has the relative congruence extension

[2]Our formulation of Theorem 3.17 is slightly different from the one of [5, Thm. 3.4]. However, the difference is insubstantial and amounts to the fact that in [5] the class K_{RFSI} is defined as $\mathsf{K}^*_{\text{RFSI}}$.

property by assumption, there exists $\eta \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{C}{\upharpoonright}_{\mathscr{L}_{\mathsf{K}}})$ such that $\theta = \eta{\upharpoonright}_A$. Recall from [4, Prop. 12.13] that $\boldsymbol{C} \in \mathsf{K}[\mathscr{L}_{\mathcal{F}}]$ implies $\mathsf{Con}_{\mathsf{M}}(\boldsymbol{C}) = \mathsf{Con}_{\mathsf{K}}(\boldsymbol{C}{\upharpoonright}_{\mathscr{L}_{\mathsf{K}}})$, whence $\eta \in \mathsf{Con}_{\mathsf{M}}(\boldsymbol{C})$. This yields $\eta{\upharpoonright}_B \in \mathsf{Con}_{\mathsf{M}}(\boldsymbol{B})$ and $\theta = (\eta{\upharpoonright}_B){\upharpoonright}_A$ because $\boldsymbol{A} \leqslant \boldsymbol{B} \leqslant \boldsymbol{C}$ and $\theta = \eta{\upharpoonright}_A$. Hence, we are done letting $\phi = \eta{\upharpoonright}_B$. ⊠

Proposition 3.19. *For every $n \geqslant 3$ the variety $\mathsf{B}(n)$ has the congruence extension property.*

Proof. We recall that every variety of Heyting algebras has the congruence extension property. In particular, $\mathbb{V}(\boldsymbol{A}_n)$ has the congruence extension property for every $n \geqslant 3$. Therefore, Proposition 3.18 yields that the pp expansion $\mathsf{B}(n)$ of $\mathbb{V}(\boldsymbol{A}_n)$ has the congruence extension property. ⊠

Proposition 3.20. *For every $n \geqslant 3$ the variety $\mathsf{B}(n)$ has the amalgamation property.*

Proof. Recall from Proposition 3.19 that the variety $\mathsf{B}(n)$ has the congruence extension property. Moreover, $\mathsf{B}(n)_{\text{FSI}}$ is closed under subalgebras by Proposition 3.10. Therefore, in view of Theorem 3.17, in order to prove that $\mathsf{B}(n)$ has the amalgamation property, it suffices to show that $\mathsf{B}(n)^*_{\text{FSI}}$ has the amalgamation property. To this end, consider a pair of embeddings $h_1 \colon \boldsymbol{A} \to \boldsymbol{B}$ and $h_2 \colon \boldsymbol{A} \to \boldsymbol{C}$ with $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathsf{B}(n)^*_{\text{FSI}}$. We need to find a pair of embeddings $g_1 \colon \boldsymbol{B} \to \boldsymbol{D}$ and $g_2 \colon \boldsymbol{C} \to \boldsymbol{D}$ with $\boldsymbol{D} \in \mathsf{B}(n)^*_{\text{FSI}}$ such that $g_1 \circ h_1 = g_2 \circ h_2$.

We have two cases depending on whether $\boldsymbol{A}$ is trivial or nontrivial. First, suppose that $\boldsymbol{A}$ is trivial. As $\mathsf{B}(n)_{\text{FSI}}$ is closed under subalgebras by Proposition 3.10 and finitely subdirectly irreducible algebras are nontrivial, we obtain that no member of $\mathsf{B}(n)_{\text{FSI}}$ has a trivial subalgebra. Since $\boldsymbol{A}$ embeds into $\boldsymbol{B}$ and $\boldsymbol{C}$, this yields $\boldsymbol{B}, \boldsymbol{C} \notin \mathsf{B}(n)_{\text{FSI}}$. Therefore, $\boldsymbol{B}$ and $\boldsymbol{C}$ are trivial because $\boldsymbol{B}, \boldsymbol{C} \in \mathsf{B}(n)^*_{\text{FSI}}$. Consequently, $\boldsymbol{A}, \boldsymbol{B}$, and $\boldsymbol{C}$ are all trivial and the embeddings $h_1 \colon \boldsymbol{A} \to \boldsymbol{B}$ and $h_2 \colon \boldsymbol{A} \to \boldsymbol{C}$ are isomorphisms. Therefore, we may assume that $\boldsymbol{A} = \boldsymbol{B} = \boldsymbol{C}$ and that h_1 and h_2 are the identity map i on A. Hence, letting $\boldsymbol{D} = \boldsymbol{A}$ and $g_1 = g_2 = i$, we are done.

Next we consider the case where $\boldsymbol{A}$ is nontrivial. Since $\boldsymbol{A}$ embeds into $\boldsymbol{B}$ and $\boldsymbol{C}$, we obtain that $\boldsymbol{B}$ and $\boldsymbol{C}$ are also nontrivial. Together with $\boldsymbol{B}, \boldsymbol{C} \in \mathsf{B}(n)^*_{\text{FSI}}$, this yields $\boldsymbol{B}, \boldsymbol{C} \in \mathsf{B}(n)_{\text{FSI}}$. Recall from Proposition 3.10 that $\mathsf{B}(n)_{\text{FSI}} = \mathbb{IS}(\boldsymbol{B}_n)$, whence $\boldsymbol{B}, \boldsymbol{C} \in \mathbb{IS}(\boldsymbol{B}_n)$. Therefore, we may assume that $\boldsymbol{B} = \boldsymbol{C} = \boldsymbol{B}_n$ and that h_1 and h_2 are embeddings of $\boldsymbol{A}$ into $\boldsymbol{B}_n$. By Proposition 3.13(ii) there exists $i \in \mathsf{aut}(\boldsymbol{B}_n)$ such that $h_1 = i \circ h_2$. Let $\boldsymbol{D} = \boldsymbol{B}_n$, $g_2 = i$, and g_1 the identity map on B_n. Clearly, $g_1, g_2 \colon \boldsymbol{B}_n \to \boldsymbol{B}_n$ are embeddings such that $g_1 \circ h_1 = h_1 = i \circ h_2 = g_2 \circ h_2$. ⊠

We are now ready to prove Theorem 3.9.

Proof. Recall that $\mathsf{B}(n)$ is a pp expansion of $\mathbb{V}(\boldsymbol{A}_n)$. Moreover, since $\mathbb{V}(\boldsymbol{A}_n)$ has the congruence extension property, we can apply [4, Thm. 12.4(ii)], obtaining that $\mathsf{B}(n)$ is congruence preserving. Hence, by [4, Thm. 11.6] it will be enough to prove that $\mathsf{B}(n)$ has the strong epimorphism surjectivity property. Recall from Propositions 3.10 and 3.20 that $\mathsf{B}(n)$ is an arithmetical variety with the amalgamation property. Therefore, in view of [4, Cor. 7.16], it will be enough to show that every $\boldsymbol{C} \in \mathsf{B}(n)_{\text{FSI}}$ lacks proper $\mathsf{B}(n)$-epic subalgebras. To this end, consider $\boldsymbol{A} \leqslant \boldsymbol{C} \in \mathsf{B}(n)_{\text{FSI}}$ with $\boldsymbol{A} \leqslant \boldsymbol{C}$ proper. Then there exists $b \in C - A$. Moreover, we may assume that $\boldsymbol{C} \leqslant \boldsymbol{B}_n$ by Proposition 3.10, whence $\boldsymbol{A} \leqslant \boldsymbol{C} \leqslant \boldsymbol{B}_n$.

Let i be the identity map on $\boldsymbol{B}_n$. As $i \in \mathsf{end}(\boldsymbol{B}_n)$ and $b \in C$, to conclude the proof, it will be enough to find some $h \in \mathsf{end}(\boldsymbol{B}_n)$ such that $h{\restriction}_A = i{\restriction}_A$ and $h(b) \neq i(b)$. For, by considering the restrictions of h and i to $\boldsymbol{C} \leqslant \boldsymbol{B}_n$, we obtain that $\boldsymbol{A} \leqslant \boldsymbol{C}$ is not $\mathsf{B}(n)$-epic, as desired.

We have two cases: either $e \notin A$ or $e \in A$. First, suppose that $e \notin A$. Since $\boldsymbol{A} \leqslant \boldsymbol{B}_n$, we have $\boldsymbol{A}{\restriction}_{\mathscr{L}} \leqslant (\boldsymbol{B}_n){\restriction}_{\mathscr{L}} = \boldsymbol{A}_n$. Together with $e \notin A$ and (3), this yields $A = \{0,1\}$. Then $0 < b$ because $b \notin A$. Let $a \in \mathsf{at}_{\boldsymbol{B}_n}(b)$ and consider the map $h\colon \boldsymbol{B}_n \to \boldsymbol{B}_n$ defined for every $c \in B_n$ as

$$h(c) = \begin{cases} 1 & \text{if } a \leqslant c; \\ 0 & \text{if } a \nleqslant c. \end{cases}$$

Since $h \in \mathsf{end}(\boldsymbol{A}_n)$ and $\boldsymbol{B}_n = \boldsymbol{A}_n[\mathscr{L}_{\mathcal{F}_n}]$, from [4, Prop. 9.5] it follows that $h \in \mathsf{end}(\boldsymbol{B}_n)$. Moreover, $a \in \mathsf{at}_{\boldsymbol{B}_n}(b)$ and the definition of h imply $h(b) = 1$. Then $h(b) \neq b$ because $b \notin A = \{0,1\}$. Thus, $h, i\colon \boldsymbol{B}_n \to \boldsymbol{B}_n$ are homomorphisms such that $h(b) \neq b = i(b)$ and $h{\restriction}_A = i{\restriction}_A$ (the latter because $A = \{0,1\}$ and both h and i preserve the constants 0 and 1).

Lastly, we consider the case where $e \in A$. As $\boldsymbol{A} \leqslant \boldsymbol{C}$ is proper and $\boldsymbol{C} \leqslant \boldsymbol{B}_n$, there exists $b \in C - (A \cup \{e\}) \subseteq B_n - (A \cup \{e\})$. By Proposition 3.13(i) there also exists $h \in \mathsf{aut}(\boldsymbol{B}_n)$ such that $b \neq h(b)$ and $a = h(a)$ for every $a \in A$. Thus, $h, i\colon \boldsymbol{B}_n \to \boldsymbol{B}_n$ are homomorphisms such that $h{\restriction}_A = i{\restriction}_A$ and $h(b) \neq b = i(b)$. ⊠

Lastly, we prove Theorem 3.3. Notice that this concludes the proof of Theorem 3.2.

Proof. As $\mathsf{B}(n)$ is a congruence preserving Beth companion of $\mathbb{V}(\boldsymbol{A}_n)$ by Theorem 3.9, it will be enough to show that $\mathsf{B}(n)$ is not equational. Suppose the contrary, with a view to contradiction. Then let a be an atom of $\boldsymbol{B}_n$ and consider $\boldsymbol{C} = \mathsf{Sg}^{\boldsymbol{B}_n}(a)$. The following is an immediate consequence of the definition of $\boldsymbol{C}$.

Claim 3.21. *The universe of $\boldsymbol{C}$ is $\{0, a, \neg a, e, 1\}$. Moreover, the Heyting algebra reduct of $\boldsymbol{C}$ is isomorphic to $\boldsymbol{A}_2$ with minimum 0, maximum 1, second largest element e, and atoms a and $\neg a$.*

As a is an atom of $\boldsymbol{B}_n$ and $\boldsymbol{A}_n$ shares its bounded lattice reduct with $\boldsymbol{B}_n$, the number of atoms of $\boldsymbol{A}_n$ below a is 1. Since $\boldsymbol{A}_n$ has $n \geqslant 3$ atoms, from (7) it follows that the number of atoms of $\boldsymbol{A}_n$ below $\neg a$ is $n - 1 \geqslant 3 - 1 \geqslant 2$. Therefore, from Corollary 3.8 it follows that $\ell_{f_{1,n}}^{\boldsymbol{B}_n}(a) = 1$ and $\ell_{f_{1,n}}^{\boldsymbol{B}_n}(\neg a) = e$. As $\boldsymbol{C} \leqslant \boldsymbol{B}_n$, we obtain

$$\ell_{f_{1,n}}^{\boldsymbol{C}}(a) = 1 \quad \text{and} \quad \ell_{f_{1,n}}^{\boldsymbol{C}}(\neg a) = e. \tag{27}$$

Recall from the assumptions that $\mathsf{B}(n)$ is equational. Therefore, by [4, Rem. 11.12(vi)] it is faithfully term equivalent relative to $\mathbb{V}(\boldsymbol{A}_n)$ to a Beth companion M of $\mathbb{V}(\boldsymbol{A}_n)$ induced by implicit operations defined by conjunctions of equations. By [4, Thm. 10.4] the Beth companion M is of the form $\mathbb{V}(\boldsymbol{A}_n)[\mathscr{L}_{\mathcal{F}}^*]$ with $\mathcal{F} \subseteq \mathsf{ext}_{eq}(\mathbb{V}(\boldsymbol{A}_n))$ and $\mathscr{L}_{\mathcal{F}}^*$ an $\mathcal{F}$-expansion of the language $\mathscr{L}$ of Heyting algebras. Furthermore, recall that $\mathsf{B}(n)$ is a variety by Proposition 3.10. Therefore, from [4, Rem. 11.12(v)] it follows that the class $\mathbb{V}(\boldsymbol{A}_n)[\mathscr{L}_{\mathcal{F}}^*]$ is also a variety.

Let τ and ρ be the maps witnessing the fact that $\mathsf{B}(n)$ and $\mathbb{V}(\boldsymbol{A}_n)[\mathscr{L}_{\mathcal{F}}^*]$ are faithfully term equivalent relative to $\mathbb{V}(\boldsymbol{A}_n)$. We may assume that for every $\boldsymbol{D} \in \mathsf{B}(n)$,

$$\tau(\boldsymbol{D}) \in \mathbb{V}(\boldsymbol{A}_n)[\mathscr{L}_{\mathcal{F}}^*] \quad \text{and} \quad \boldsymbol{D}{\restriction}_{\mathscr{L}} = \tau(\boldsymbol{D}){\restriction}_{\mathscr{L}}. \tag{28}$$

As $\boldsymbol{C} \leqslant \boldsymbol{B}_n \in \mathsf{B}(n)$ and $\mathsf{B}(n)$ is a variety by Proposition 3.10, we have $\boldsymbol{C} \in \mathsf{B}(n)$. Then $\tau(\boldsymbol{C}) \in \mathbb{V}(\boldsymbol{A}_n)[\mathscr{L}^*_{\mathcal{F}}]$ by the left hand side of (28). Consequently, there exists $\boldsymbol{D} \in \mathbb{V}(\boldsymbol{A}_n)$ such that $\boldsymbol{D}[\mathscr{L}^*_{\mathcal{F}}]$ is defined and $\tau(\boldsymbol{C}) = \boldsymbol{D}[\mathscr{L}^*_{\mathcal{F}}]$. Together with the right hand side of (28), this yields

$$\boldsymbol{C}{\restriction}_{\mathscr{L}} = \tau(\boldsymbol{C}){\restriction}_{\mathscr{L}} = \boldsymbol{D}[\mathscr{L}^*_{\mathcal{F}}]{\restriction}_{\mathscr{L}} = \boldsymbol{D}.$$

In view of the above display, $\boldsymbol{D}$ is the Heyting algebra reduct of $\boldsymbol{C}$ and, therefore, is isomorphic to $\boldsymbol{A}_2$ with atoms a and $\neg a$ by Claim 3.21. This allows us to apply Proposition 3.12 to the permutation $\sigma \colon \mathsf{at}(\boldsymbol{D}) \to \mathsf{at}(\boldsymbol{D})$ that switches a and $\neg a$, thus obtaining an automorphism $\sigma^* \colon \boldsymbol{D} \to \boldsymbol{D}$ with

$$\sigma^*(a) = \neg a \quad \text{and} \quad \sigma^*(1) = 1. \tag{29}$$

Moreover, from $\tau(\boldsymbol{C}) = \boldsymbol{D}[\mathscr{L}^*_{\mathcal{F}}]$ it follows that $\boldsymbol{C} = \rho\tau(\boldsymbol{C}) = \rho(\boldsymbol{D}[\mathscr{L}^*_{\mathcal{F}}])$. Together with (27), this implies

$$\rho(\ell_{f_{1,n}})^{\boldsymbol{D}[\mathscr{L}^*_{\mathcal{F}}]}(a) = 1 \quad \text{and} \quad \rho(\ell_{f_{1,n}})^{\boldsymbol{D}[\mathscr{L}^*_{\mathcal{F}}]}(\neg a) = e. \tag{30}$$

Recall from [4, Prop. 10.22(ii)] that there exists $g \in \mathsf{ext}_{pp}(\mathbb{V}(\boldsymbol{A}_n))$ such that

$$\rho(\ell_{f_{1,n}})^{\boldsymbol{D}[\mathscr{L}_{\mathcal{F}^*}]} = g^{\boldsymbol{D}}. \tag{31}$$

Together with the left hand side of (30), this yields $g^{\boldsymbol{D}}(a) = 1$. As the implicit operation g is preserved by homomorphisms, we can apply the automorphism σ^* of $\boldsymbol{D}$ in (29) to deduce

$$g^{\boldsymbol{D}}(\neg a) = g^{\boldsymbol{D}}(\sigma^*(a)) = \sigma^*(g^{\boldsymbol{D}}(a)) = \sigma^*(1) = 1$$

and, therefore, $\rho(\ell_{f_{1,n}})^{\boldsymbol{D}[\mathscr{L}_{\mathcal{F}^*}]}(\neg(a)) = 1$ by (31). Since $1 \neq e$, this contradicts the right hand side of (30). Hence, we conclude that $\mathsf{B}(n)$ is a congruence preserving Beth companion of $\mathbb{V}(\boldsymbol{A}_n)$ that is not equational. ⊠

References

[1] R. Balbes and P. Dwinger. *Distributive lattices*. University of Missouri Press, Columbia, Mo., 1974.

[2] C. Bergman. *Universal Algebra: Fundamentals and Selected Topics*. Chapman & Hall Pure and Applied Mathematics. Chapman and Hall/CRC, 2011.

[3] S. Burris and H. P. Sankappanavar. *A Course in Universal Algebra*. 2012. The millennium edition, available online.

[4] L. Carai, M. Kurtzhals, and T. Moraschini. The theory of implicit operations. Available on arXiv, 2025.

[5] W. Fussner and G. Metcalfe. Transfer theorems for finitely subdirectly irreducible algebras. *J. Algebra*, 640:1–20, 2024.

[6] S. Givant and P. Halmos. *Introduction to Boolean algebras*. Undergraduate Texts in Mathematics. Springer, New York, 2009.

[7] V. A. Gorbunov. *Algebraic theory of quasivarieties*. Siberian School of Algebra and Logic. Consultants Bureau, New York, 1998. Translated from the Russian.

[8] G. Grätzer and H. Lakser. The structure of pseudocomplemented distributive lattices. II. Congruence extension and amalgamation. *Trans. Amer. Math. Soc.*, 156:343–358, 1971.

[9] T. Hosoi and H. Ono. The intermediate logics on the second slice. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 17:457–461, 1970.

[10] L. L. Maksimova. Craig's theorem in superintuitionistic logics and amalgamable varieties. *Algebra i Logika*, 16(6):643–681, 741, 1977.

[11] L. L. Maksimova. Intuitionistic logic and implicit definability. *Ann. Pure Appl. Logic*, 105(1-3):83–102, 2000.

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